Efficient Spectral Methods for Learning Mixture Models

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Based on joint works with Munther Dahleh, Rong Ge, Sham Kakade, Greg Valiant.

Learning

Data generation

θ → $X(\theta)$

θ ← $X(\theta)$

Learning

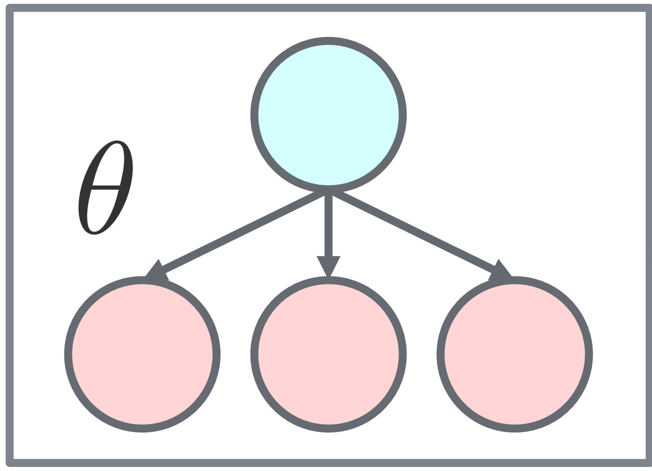
- Infer about the underlying rule θ
  (Estimation, approximation, property testing, optimization of f(θ))

- Challenge:
  Exploit our prior for structure of the underlying θ
  to design fast algorithm that uses as few as possible data X
  to achieve the target accuracy in learning θ

Computation Complexity

Sample Complexity $\uparrow \dim(\theta)$

Learning Mixture Models



Samples →

Hidden variable $H \in \{1, \dots, K\}$

Observed variables $X = (X_1, X_2, \dots, X_M) \in \mathcal{X}$

- Marginal distribution of the observables is a superposition of simple distributions

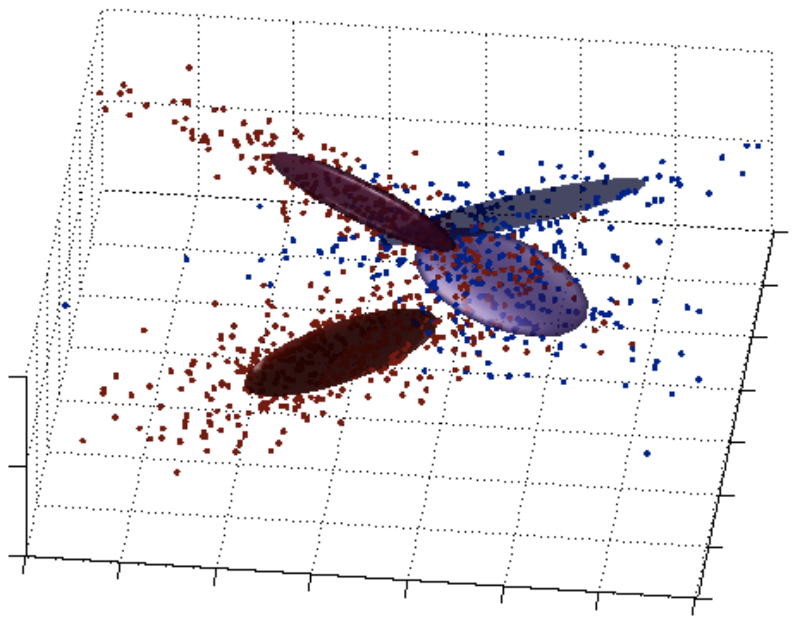
$$\Pr(X) = \sum_{k=1}^{K} \underbrace{\Pr(H = k)} \cdot \underbrace{\Pr(X | H = k)}$$

θ = (#mixture components, mixing weights, conditional probabilities)

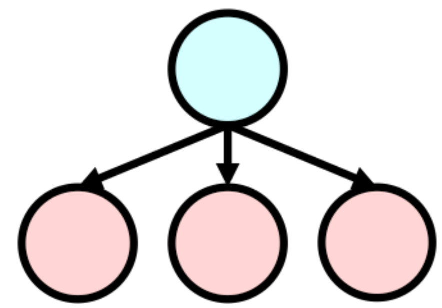
- Given N i.i.d. samples of observable variables, estimate the model parameters $\widehat{\theta}$

$$\left\| \widehat{\theta} - \theta \right\| \leq \epsilon$$

Examples of Mixture Models



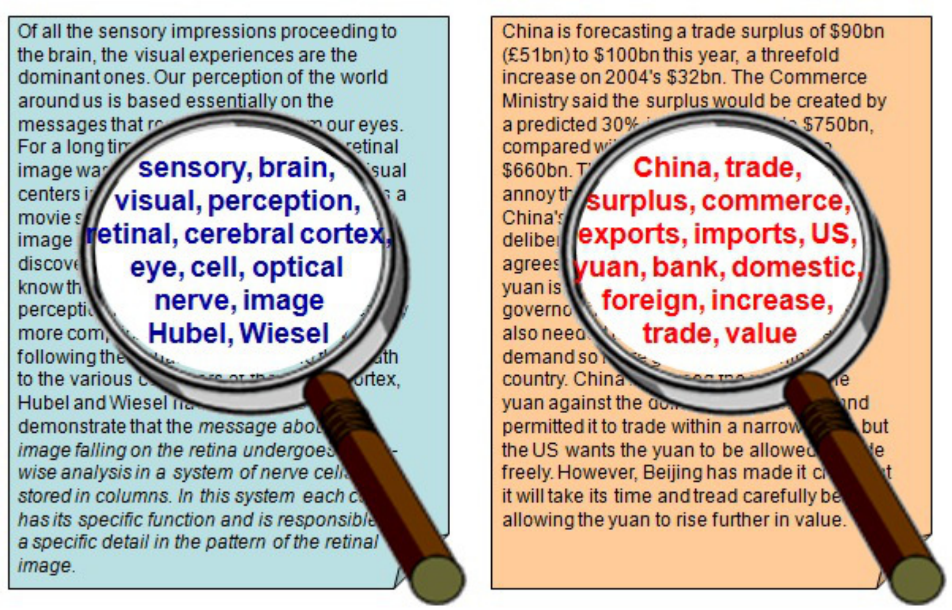
Gaussian Mixtures (GMMs)



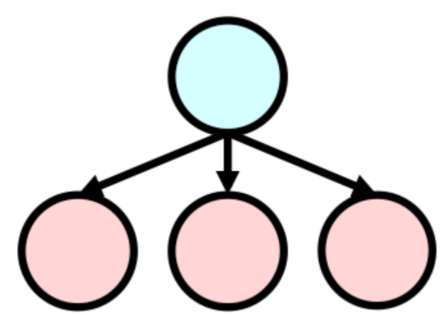
Cluster

θ

data points in space



Topic Models (Bag of Words)

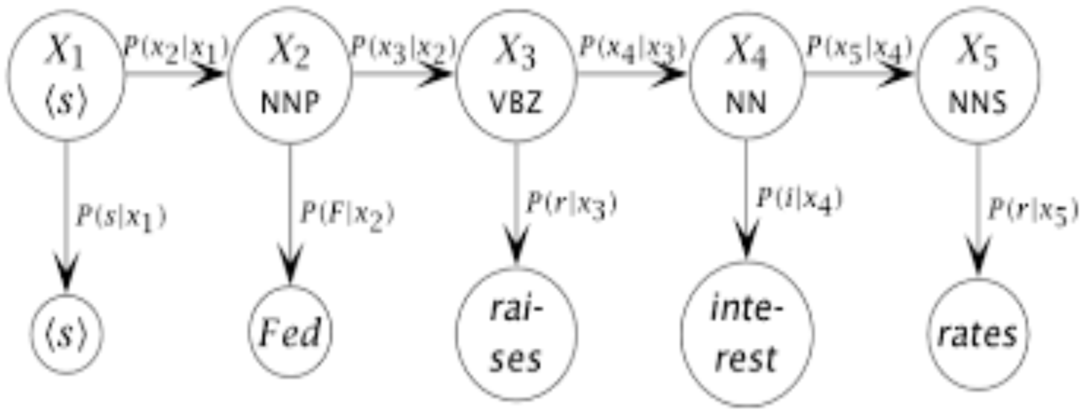


Topic

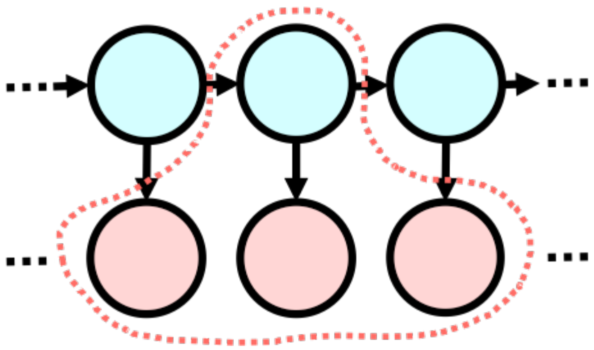
θ

words
in each document

Examples of Mixture Models



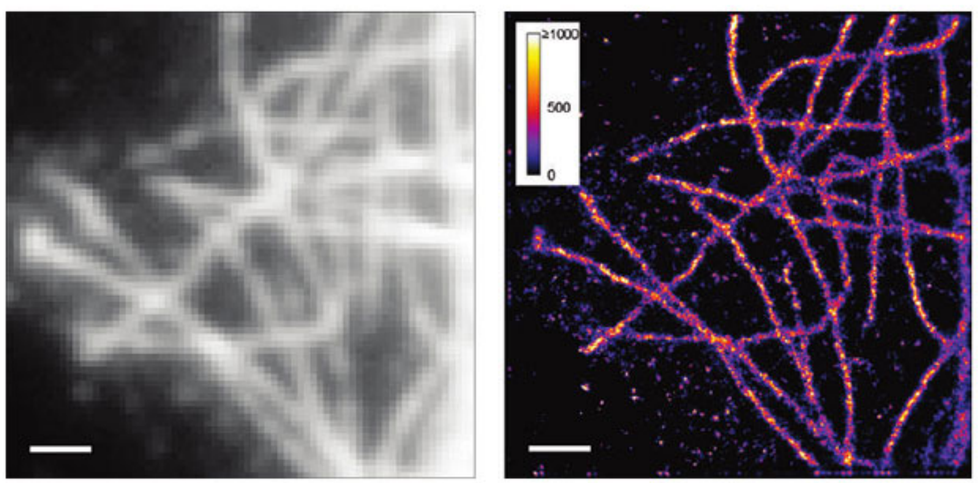
Hidden Markov Models (HMM)

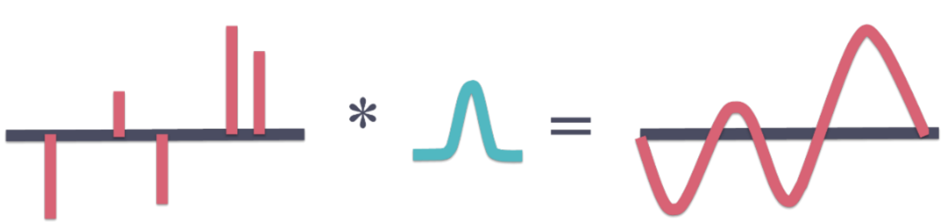


Current state

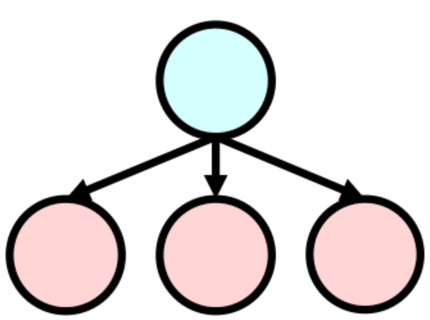
θ

Past, current, future observations





Super-Resolution

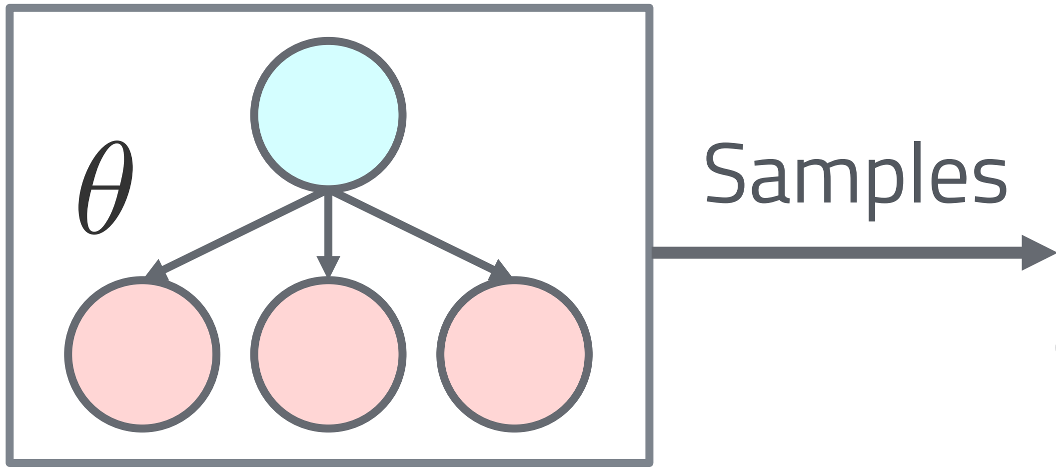


Source

θ

Complex sinusoids

Learning Mixture Models



Hidden variable $H \in \{1, \ldots, K\}$

Observed variables $X = (X_1, X_2, \ldots, X_M) \in \mathcal{X}$

- Given N i.i.d. samples of observable variables, estimate the model parameters $\widehat{\theta}$

What do we know about sample complexity and computation complexity ?

- There exist exponential lower bounds for sample complexity (worst case analysis)
- Maximum Likelihood Estimation is non-convex optimization (EM is heuristics)

Challenges in Learning Mixture Models

- There exist exponential lower bounds for sample complexity (worst case analysis)

  Can we learn "**non-worst-cases**" with provably efficient algorithms ?

- Maximum Likelihood Estimation is non-convex optimization (EM is heuristics)

  Can we achieve **statistical efficiency** with **tractable** computation?

Contribution / Outline of the talk

- There exist exponential lower bounds for sample complexity (worst case analysis)

  Can we learn "**non-worst-cases**" with provably efficient algorithms ?

✓ Spectral algorithms for learning GMMs, HMMs, Super-resolution
✓ Go beyond worst cases with **randomness** in analysis and algorithm

- Maximum Likelihood Estimation is non-convex optimization (EM is heuristics)

  Can we achieve **statistical efficiency** with **tractable** computation?

✓ Denoising low rank probability matrix with **linear** sample complexity (minmax optimal)

Paradigm of Spectral Algorithms for Learning

Mixture models

θ

Samples

Mixture of conditional distributions

$X(\theta)$

N = ∞

Compute statistics:
e.g. Moments,
Marginal probability

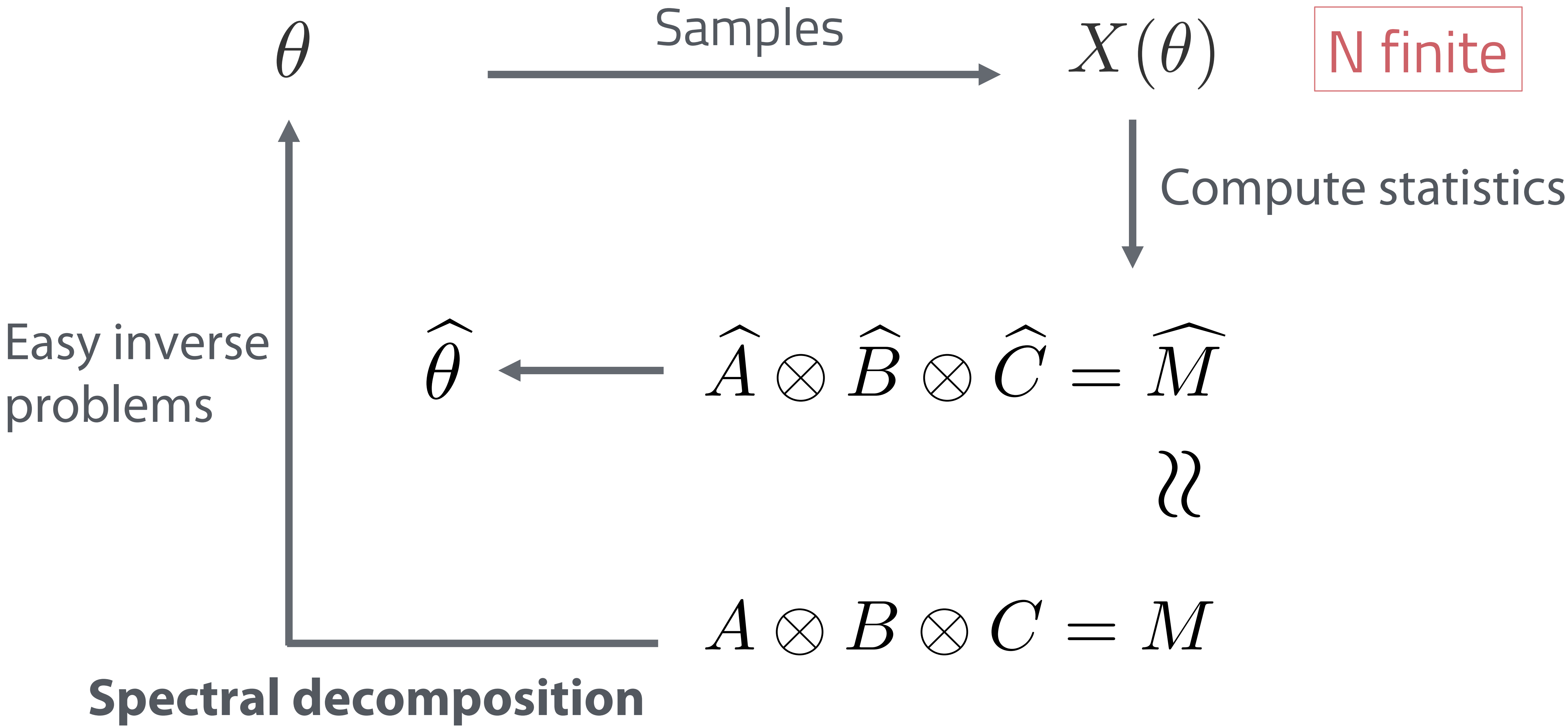
$$A \otimes B \otimes C = M$$

Mixture of rank-one matrices/tensors

Spectral decomposition to separate the mixtures

Easy inverse problems

Paradigm of Spectral Algorithms for Learning



- ✓ PCA, Spectral clustering, Subspace system ID,… fit into this paradigm
- ✓ Problem specific algorithm design (what statistics M to use?)
  analysis (is the spectral decomposition stable?)

PART 1

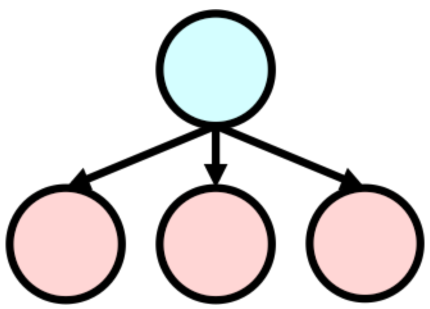
Provably efficient spectral algorithms for learning mixture models

1.1 Learn GMMs: Go beyond worst cases by **smoothed analysis**

1.2 Learn HMMs: Go beyond worst cases by **generic analysis**

1.3 Super-resolution: Go beyond worst cases by **randomized algorithm**

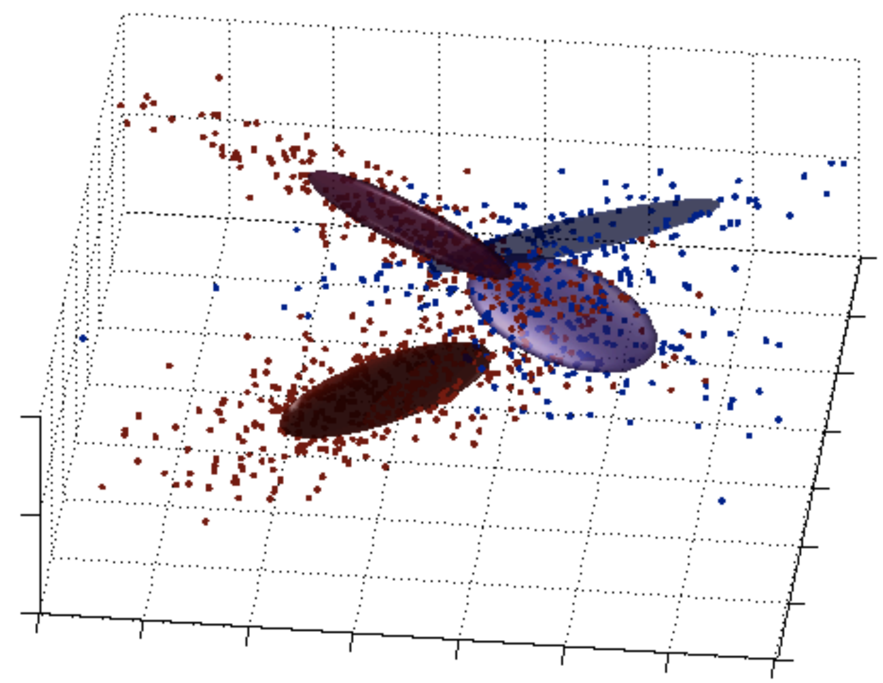
1.1 Learn GMMs: Setup



Cluster $[k]$

θ

M-dim data points $x \in \mathbb{R}^n$



mixture of k multivariate Gaussians → data points in n-dimensional space

Model Parameters: weights w_i means $\mu^{(i)}$ covariance matrices $\Sigma^{(i)}$

$$x = \mathcal{N}(\mu^{(i)}, \Sigma^{(i)}), \quad i \sim w_i$$

1.1 Learn GMMs: Prior Works

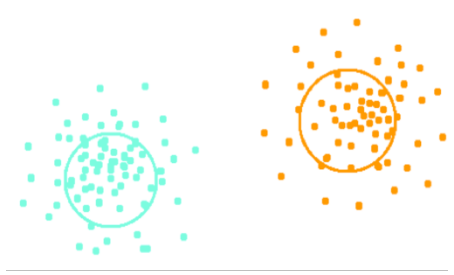
- **General case**

  Moment matching method [Moitra&Valiant] [Belkin&Sinha] $Poly(n, e^{O(k)^k})$

- **With restrictive assumptions on model parameters**

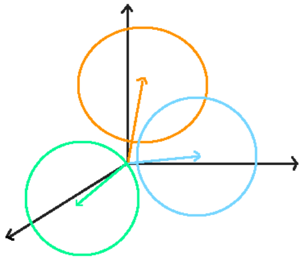
  - ✓ Mean vectors are well-separated

    Pair wise clustering [Dasgupta]…[Vempala&Wang] $Poly(n, k)$



  - ✓ Mean vectors of spherical Gaussians are linearly independent

    Moments tensor decomposition [Hsu&Kakade] $Poly(n, k)$



1.1 Learn GMMs: Worst case lower bound

Can we learn **every** GMM instance to target accuracy
in poly runtime and using poly samples?

No!

Exponential dependence in k for worst cases. [Moitra&Valiant]

Can we learn **most** GMM instances with **poly** algorithm?
without restrictive assumptions on model parameters

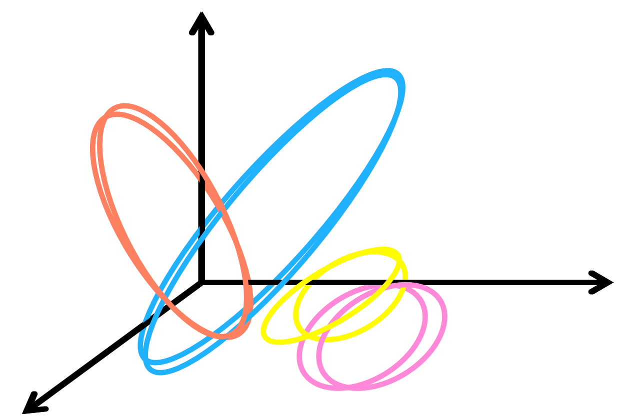
Yes!

Worst cases are not everywhere, and we can handle “non-worst-cases”

1.1 Learn GMMs: Smoothed Analysis Framework

Escape from the worst cases

For an arbitrary instance θ



Nature perturbs the parameters with a small amount (**ρ**) of noise $\widetilde{\theta}$

Observe data generated by $\widetilde{\theta}$, design and analyze algorithm for $\widetilde{\theta}$

Hope: With high probability over nature's perturbation, any arbitrary instance escapes from the degenerate cases, and becomes well conditioned.

- ✓ Bridge worst case and average case algo analysis [Spielman&Teng]
- ✓ A stronger notion than generic analysis

Our Goal:

Given samples from perturbed GMM, learn the perturbed parameters with negligible failure probability over nature's perturbation

Main Results

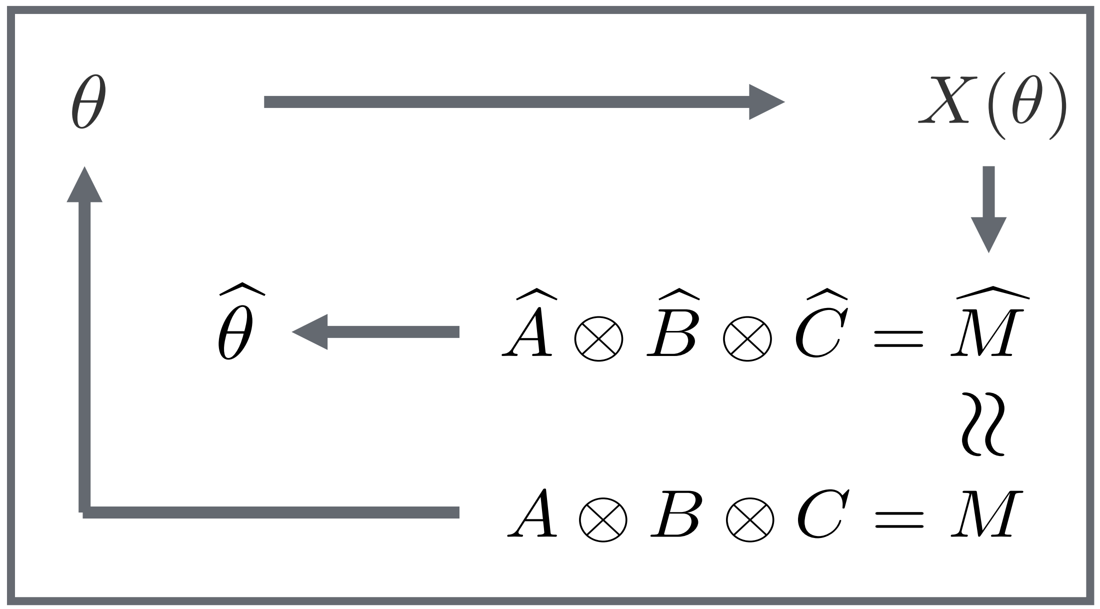
- Our algorithm learns the GMM parameters up to accuracy ε
  - ✓ With fully polynomial time and sample complexity $Poly(n, k, 1/\epsilon)$
  - ✓ Assumption: data in high enough dimension $n = \Omega(k^2)$
  - ✓ Under smoothed analysis: works with negligible failure probability

"Learning Mixture of Gaussians in High dimensions" R. Ge, **H**, S. Kakade (STOC 2015)

1.1 Learn GMMs: Algorithmic Ideas

Method of moments: match 4-th and 6-th order moments $M_4 \; M_6$

Key challenge: Moment tensors are not of low rank, but they have special structures



$$M_4 = \mathbb{E}[x^{\otimes 4}]$$
$$M_6 = \mathbb{E}[x^{\otimes 6}]$$

$$X_4 = \sum_{i=1}^{k} \Sigma^{(i)} \otimes \Sigma^{(i)},$$
$$X_6 = \sum_{i=1}^{k} \Sigma^{(i)} \otimes \Sigma^{(i)} \otimes \Sigma^{(i)}.$$

← Structured linear projection

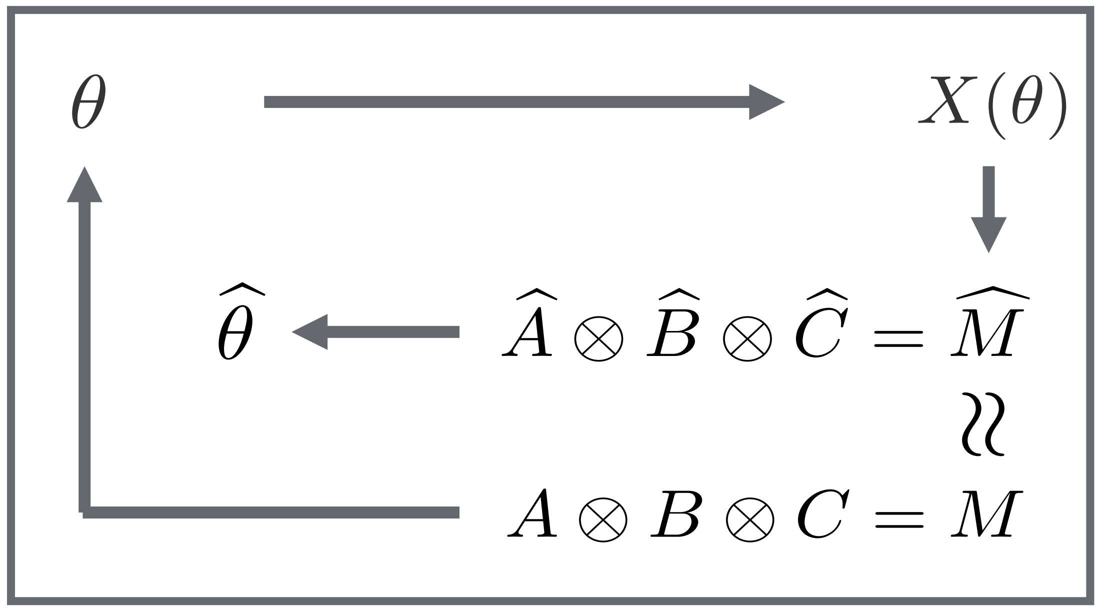
$$M_4 = \mathcal{F}_4(X_4)$$
$$M_6 = \mathcal{F}_6(X_6)$$

- Moment tensors are structured linear projections of desired low rank tensors
- Delicate algorithm to invert the structured linear projections

1.1 Learn GMMs: Algorithmic Ideas

Method of moments: match 4-th and 6-th order moments M_4 M_6

Key challenge: Moment tensors are not of low rank, but they have special structures



$$M_4 = \mathbb{E}[x^{\otimes 4}]$$
$$M_6 = \mathbb{E}[x^{\otimes 6}]$$

Why "high dimension n" & "smoothed analysis" help us to learn?

- ✓ We have many moment matching constraints with only low order moments

  # free parameters $\Omega(kn^2)$ < #6-th moments $\Omega(n^6)$

- ✓ The randomness in nature's perturbation makes matrices/tensors well-conditioned

  Gaussian matrix $X \in \mathbb{R}^{n\times m}$, with prob at least $1 - O(\epsilon^n)$ $\sigma_m(X) \geq \epsilon\sqrt{n}$.

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Provably efficient spectral algorithms for learning mixture models

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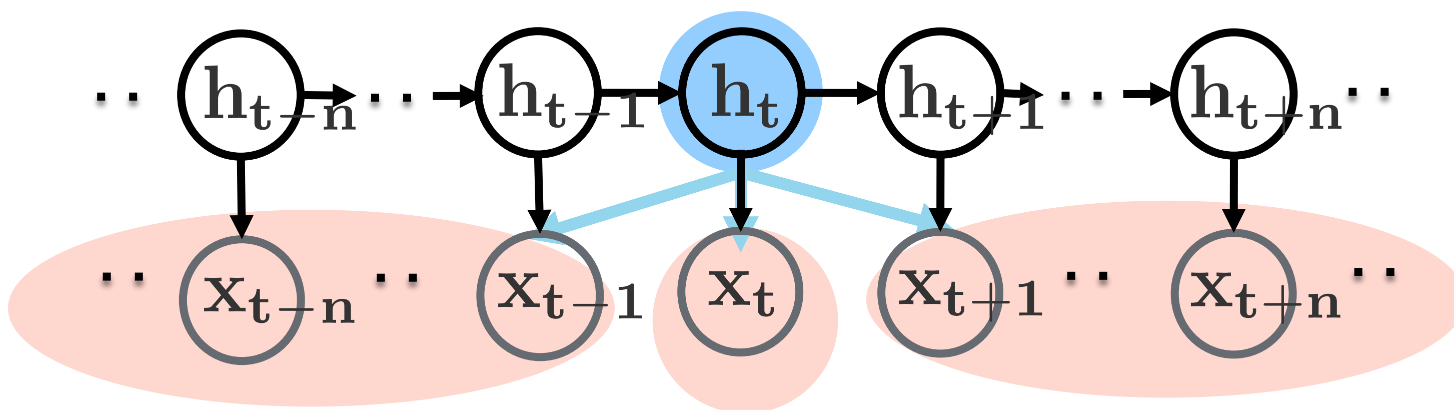
1.2 Learn HMMs: Go beyond worst cases by **generic analysis**

1.3 Super-resolution: Go beyond worst cases by **randomized algorithm**

1.2 Learn HMMs: Setup



N = 2n+1 window size

Transition probability matrix: $Q \in \mathbb{R}^{k \times k}$

Observation probabilities: $O \in \mathbb{R}^{d \times k}$

Given length-N sequences of observation, how to recover $\theta = (Q, O)$?

Our focus: How large the window size N needs to be?

1.2 Learn HMMs:

Hidden state $[k]$ Observation $[d]$ N = 2n+1 window size

Hardness Results

- HMM is not efficiently PAC learnable, under noisy parity assumption

  Construct an instance with reduction to parity of noise [Abe,Warmuth] [Kearns]

  Required window size $N = \Omega(k)$, Algorithm Complexity is $\Omega(d^k)$

Our Result

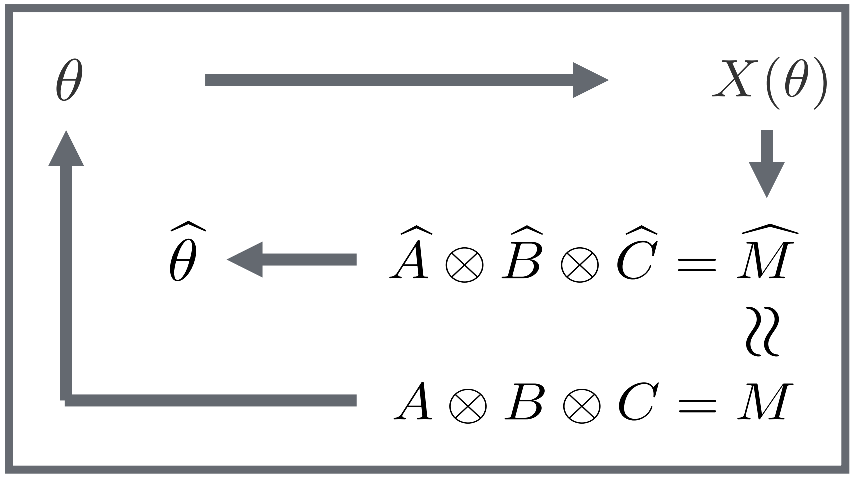
- Excluding a measure 0 set in the parameter space of $\theta = (Q, O)$

  for all most all HMM instances, the required window size is $N = \Theta(\log_d k)$

- Spectral algo achieves sample complexity and runtime both $poly(d, k)$

"Minimal Realization Problems for Hidden Markov Models"
H, R. Ge, S. Kakade, M. Dahleh (IEEE Transactions on Signal Processing, 2016)

1.2 Learn HMMs: Algorithmic idea



$$M = \Pr\left((x_{n-1},\dots,x_{-1}),x_0,(x_1,\dots,x_n)\right)$$
$$\in \mathbb{R}^{d^n\times d^n\times d}$$

1. M is a low rank tensor of rank k

$$M = A\otimes B\otimes C$$

$$A = \Pr\left(x_1,x_2,\dots,x_n \middle| h_0\right)$$
$$B = \Pr\left(x_{-1},x_{-2},\dots,x_{-n} \middle| h_0\right)$$
$$C = \Pr\left(x_0,h_0\right)$$

2. Extract Q, O from tensor factors A B

$$A = \underbrace{(O\odot(O\odot(O\odot\dots(O\odot O}_{n}\underbrace{Q)\dots)Q)Q)Q,}_{n}$$
$$B = \underbrace{(O\odot(O\odot(O\odot\dots(O\odot O}_{n}\underbrace{\widetilde{Q})\dots)\widetilde{Q})\widetilde{Q})\widetilde{Q},}_{n}$$

Key challenge:
How large window size needs to be, so that we have unique tensor decomp

Our careful generic analysis:
If $N=\Theta(\log_d k)$, worst cases all lie in a measure 0 set!

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Provably efficient spectral algorithms for learning mixture models

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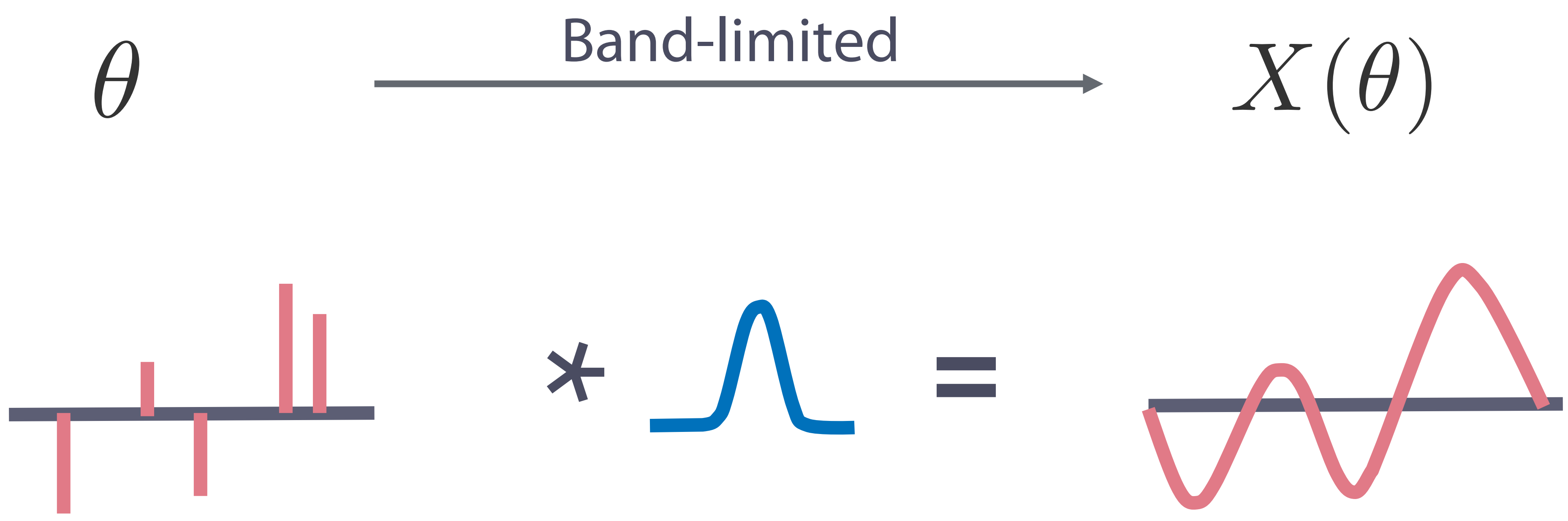
1.3 Super-resolution: Go beyond worst cases by **randomized algorithm**

1.3 Super-Resolution: Setup



Take Fourier measurement of the band-limited observation

How to recover the point sources with **coarse** measurement of the signal?

- ✓ small number of Fourier measurements
- ✓ Low cutoff frequency

1.3 Super-Resolution: Problem Formulation

- ✓ Recover point sources (a mixture of k vectors in d-dimensional space)

$$x(t) = \sum_{j=1}^{k} w_j \delta_{\mu^{(j)}}.$$

Assume minimum separation $\Delta = \min_{j \neq j'} \|\mu^{(j)} - \mu^{(j')}\|_2$

- ✓ Use bandlimited and noisy Fourier measurements.

$$\widetilde{f}(s) = \sum_{j=1}^{k} w_j e^{i\pi <\mu^{(j)}, s>} + z(s)$$

$\|s\|_\infty \leq \text{cutoff freq}$ bounded noise $|z(s)| \leq \epsilon_z, \forall s$

- ✓ Achieve target accuracy $\|\widehat{\mu}^{(j)} - \mu^{(j)}\|_2 \leq \epsilon, \forall j \in [k]$

1.3 Super-Resolution: Prior Works

$$\widetilde{f}(s) = \sum_{j=1}^{\kappa} w_j e^{i\pi<\mu^{(j)},s>} + z(s) \qquad \Delta = \min_{j\neq j'} \|\mu^{(j)} - \mu^{(j')}\|_2$$

- **1-dimensional** $\mu^{(j)}$
  - Take uniform measurements on the grid $s \in \{-N, \dots, -1, 0, 1, \dots, N\}$
  - SDP algorithm with cut-off frequency $N = \Omega(\frac{1}{\Delta})$ [Candes, Fernandez-Granda]
  - Hardness result $N > \frac{C}{\Delta}$ [Moitra]
  - One can use $k\log(k)$ random measurements to recover $2N$ measurements [Tang, Bhaskar, Shah, Recht]
- **d-dimensional** $\mu^{(j)}$
  - Mult-dim grid $s \in \{-N, \dots, -1, 0, 1, \dots, N\}^d$
  - Algorithm complexity $O\left(poly(k, \frac{1}{\Delta})\right)^d$

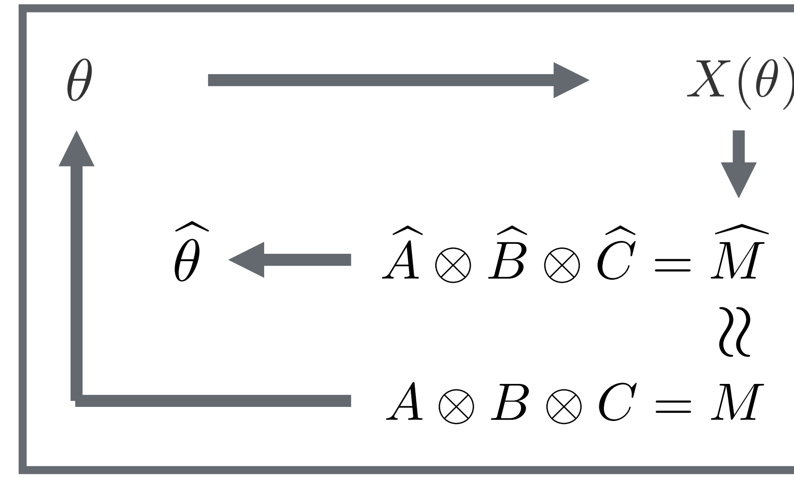
1.3 Super-Resolution: Main Result

- Our algorithm achieves stable recovery
  - ✓ using a number of $O((k+d)^2)$ Fourier measurements
  - ✓ cutoff freq of the measurements bounded by $O(1/\Delta)$
  - ✓ algorithm runtime $O((k+d)^3)$
  - ✓ algorithm works with negligible failure probability

	cutoff freq	measurements	runtime
SDP	$\frac{C_d}{\Delta_\infty}$	$(\frac{1}{\Delta_\infty})^d$	$poly((\frac{1}{\Delta_\infty})^d, k)$
MP	-	-	-
Ours	$\frac{\log(kd)}{\Delta}$	$(k\log(k)+d)^2$	$(k\log(k)+d)^2$

"Super-Resolution off the Grid" **H**, S. Kakade (NIPS 2015)

1.3 Super-Resolution: Algorithmic Idea



$$\widetilde{f}(s) = \sum_{j=1}^{k} w_j e^{i\pi <\mu^{(j)}, s>} + z(s)$$

- ✓ Tensor decomposition with measurements on **random frequencies**
- ✓ Random samples S such that F admits particular low rank decomp

$$F = V_{S'} \otimes V_{S'} \otimes (V_2 D_w),$$

(Rank-k 3-way tensor)

$d \times d \times 2$

$$V_S = \begin{bmatrix} e^{i\pi <\mu^{(1)}, s^{(1)}>} & \dots & e^{i\pi <\mu^{(k)}, s^{(1)}>} \\ e^{i\pi <\mu^{(1)}, s^{(2)}>} & \dots & e^{i\pi <\mu^{(k)}, s^{(2)}>} \\ \vdots & \dots & \vdots \\ e^{i\pi <\mu^{(1)}, s^{(m)}>} & \dots & e^{i\pi <\mu^{(k)}, s^{(m)}>} \end{bmatrix}.$$

(Vandermonde Matrix with complex nodes)

$d \times k$

- ✓ Skip intermediate step of recovering $\Omega(N^d)$ measurements on the hyper-grid
- ✓ Prony's method (Matrix-Pencil / MUSIC / …) is just choosing S deterministically

1.3 Super-Resolution: Algorithmic Idea

- ✧ Tensor decomposition with measurements on **random frequencies**

- ✧ Why **we** do not contradict the **hardness result**?

$$O((k+d)^2) \quad \text{vs} \quad O\left(poly(k, \frac{1}{\Delta})\right)^d$$

✓ If we design a **fixed** grid of S to take measurements $f(s)$
there always exists model instances such that the particular grid fails

✓ Let the locations of S be **random** (with structure for tensor decomp)
then for any model instance, algo works with high probability

PART 2

Can we achieve optimal sample complexity in a tractable way?

Estimate **low rank probability matrices** with linear sample complexity

- This problem is at the core of many spectral algorithms
- We capitalize the insights from community detection to solve it

2. Low rank matrix

Setup

$\theta \longrightarrow X$

Probability Matrix $\mathbb{B} \in \mathbb{R}_+^{M \times M}$

(distribution over M^2 outcomes)

N i.i.d. samples

(freq counts over M^2 outcomes)

$\mathbb{B}$ is of rank 2: $\mathbb{B} = pp^\top + qq^\top$

$B = \text{Poisson}(N\mathbb{B})$

$\mathbb{B}$:

.18	.14	.08	.07	.07
.14	.29	.09	.07	.10
.08	.09	.05	.40	.04
.08	.07	.04	.04	.04
.07	.10	.04	.05	.05

p, q:

p	q
.40	.15
.20	.40
.15	.15
.15	.10
.10	.20

$M = 5$

B:

5	3	2	1	1
3	4	1	0	1
2	2	1	0	1
2	1	0	1	0
1	2	1	0	0

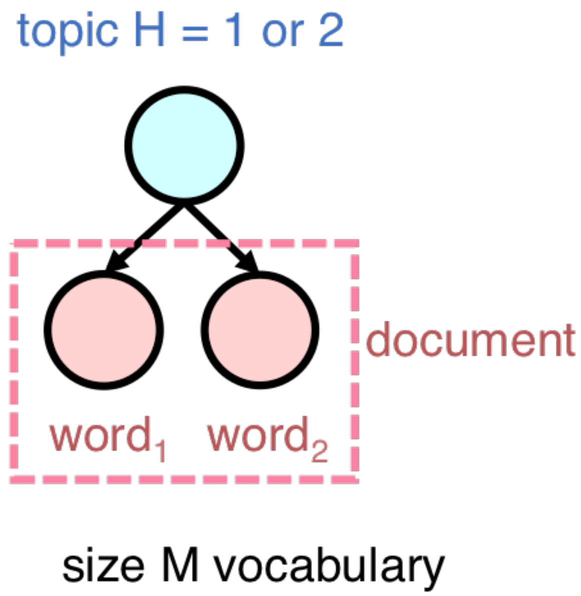
$N = 20$

Goal: find rank-2 $\widehat{B}$ such that $\|\widehat{B} - \mathbb{B}\|_1 \leq \epsilon$

N sample complexity: upper bound algorithm, lower bound

2. Low rank matrix Connection to mixture models

Topic model

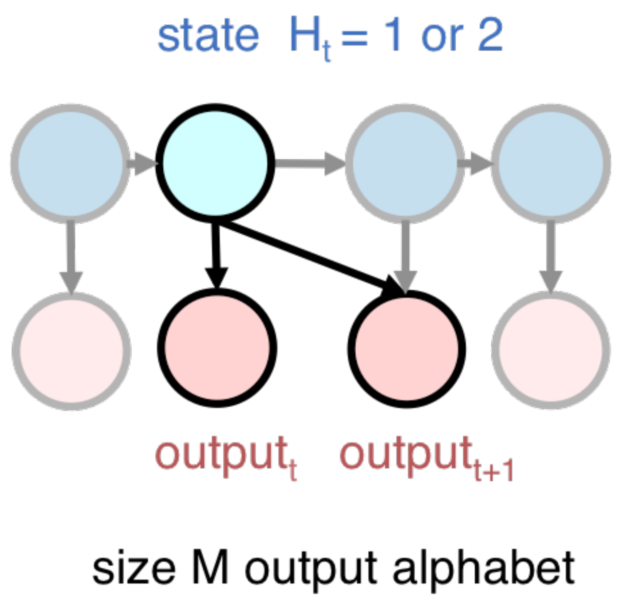


$$\Pr(\text{word}_1, \text{word}_2 | \text{topic} = T_1) = pp^\top$$

$$\Pr(\text{word}_1, \text{word}_2 | \text{topic} = T_2) = qq^\top$$

$\mathbb{B}$ joint distribution over word pairs

HMM



$$\Pr(\text{output}_1, \text{output}_2 | \text{state} = S_i) = O_i (OQ_i)^\top$$

$\mathbb{B}$ distribution of consecutive outputs

N data samples → empirical counts B → find low rank $\widehat{B}$ close to $\mathbb{B}$ → Extract parameters estimates

2. Low rank matrix Sub-Optimal Attempt

$$\theta \rightleftarrows X$$

Probability Matrix $\mathbb{B} \in \mathbb{R}_+^{M \times M}$

$\mathbb{B}$ is of rank 2: $\mathbb{B} = \rho\rho^\top + \Delta\Delta^\top$

N i.i.d. samples

$B = \mathrm{Poisson}(N\mathbb{B})$

MLE is non-convex optimization ☹

let's try something "spectral" ☺

2. Low rank matrix Sub-Optimal Attempt

$\theta \rightleftarrows X$

Probability Matrix $\mathbb{B} \in \mathbb{R}_+^{M\times M}$

N i.i.d. samples

$\mathbb{B}$ is of rank 2: $\mathbb{B} = \rho\rho^\top + \Delta\Delta^\top$

$B = \text{Poisson}(N\mathbb{B})$

$$\frac{1}{N}B = \frac{1}{N}\text{Poisson}(N\mathbb{B}) \to \mathbb{B}, \text{ as } N \to \infty$$

- Set $\widehat{B}$ to be the rank 2 truncated SVD of $\frac{1}{N}B$
- To achieve accuracy $\|\widehat{B} - \mathbb{B}\|_1 \le \epsilon$ need $N = \Omega(M^2 \log M)$
- Not sample efficient! Hopefully $N = \Omega(M)$
- Small data in practice!

  Word distribution in language has fat tail.
  More sample documents N, larger the vocabulary size M

Main Result

- Our upper bound algorithm:
  - ✓ Rank-2 estimate $\widehat{B}$ with accuracy $\|\widehat{B} - \mathbb{B}\|_1 \le \epsilon \quad \forall \epsilon > 0$
  - ✓ Using $N = O(M/\epsilon^2)$ number of sample counts
  - ✓ Runtime $O(M^3)$

  Lead to improved spectral algorithms for learning

- We prove (strong) lower bound:
  - ✓ Need a sequence of $\Omega(M)$ observations to **test** whether the sequence is i.i.d. of unif (M) or generated by a 2-state HMM

  Testing property is no easier than estimating ?!

“Recovering Structured Probability Matrices “ **H**, S. Kakade, W. Kong, G. Valiant, (submitted to STOC 2016)

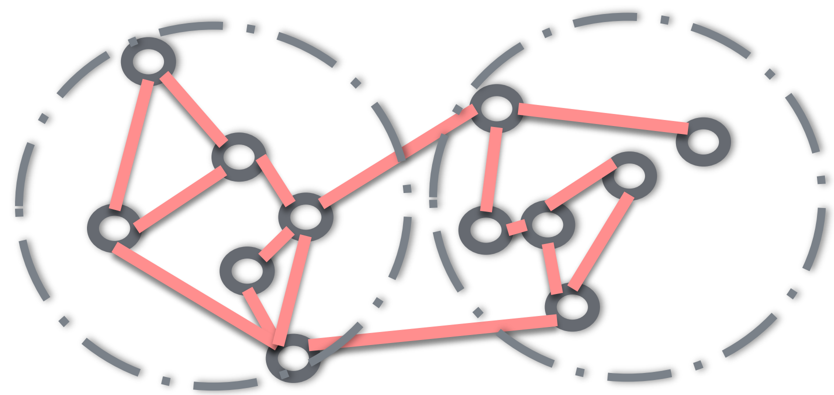
Algorithmic Idea

We capitalize the idea of community detection in sparse random network, which is a special case of our problem formulation

M nodes 2 communities

Expected connection $\mathbb{B} = pp^\top + qq^\top$

Adjacency matrix $B = \text{Bernoulli}(N\mathbb{B})$



$\mathbb{B}$:

.09	.09	.09	.02	.02	.02
.09	.09	.09	.02	.02	.02
.09	.09	.09	.02	.02	.02
.02	.02	.02	.09	.09	.09
.02	.02	.02	.09	.09	.09
.02	.02	.02	.09	.09	.09

p, q:

p	q
.30	.03
.30	.03
.30	.03
.03	.30
.03	.30
.03	.30

generate →
← estimate

B:

1	1	0	0	1	0
1	1	1	0	1	1
0	1	1	0	1	0
0	0	0	0	1	1
1	1	1	1	1	1
0	1	0	0	1	1

2. Low rank matrix

Algorithmic Idea

We capitalize the idea of community detection in sparse random network, which is a special case of our problem formulation

M nodes 2 communities

Expected connection $\mathbb{B} = pp^\top + qq^\top$

Adjacency matrix $B = \text{Bernoulli}(N\mathbb{B})$

Regularize Truncated SVD: [Le, Levina, Vershynin]

remove heavy row/column from B, run rank-2 SVD on the remaining graph

.09	.09	.09	.02	.02	.02
.09	.09	.09	.02	.02	.02
.09	.09	.09	.02	.02	.02
.02	.02	.02	.09	.09	.09
.02	.02	.02	.09	.09	.09
.02	.02	.02	.09	.09	.09

$\mathbb{B}$

p	q
.30	.03
.30	.03
.30	.03
.03	.30
.03	.30
.03	.30

generate →

← estimate

1	1	0	0	1	0
1	1	1	0	1	1
0	1	1	0	1	0
0	0	0	0	1	1
1	1	1	1	1	1
0	1	0	0	1	1

B

2. Low rank matrix — Algorithmic Idea

We capitalize the idea of community detection in sparse random network, which is a special case of our problem formulation

M nodes 2 communities	Expected connection	$\mathbb{B} = pp^\top + qq^\top$
	Adjacency matrix	$B = \text{Bernoulli}(N\mathbb{B})$

Regularize Truncated SVD: [Le, Levina, Vershynin]
remove heavy row/column from B, run rank-2 SVD on the remaining graph

$M \times M$ matrix	Probability matrix	$\mathbb{B} = \rho\rho^\top + \Delta\Delta^\top$
	Sample counts	$B = \text{Poisson}(N\mathbb{B})$

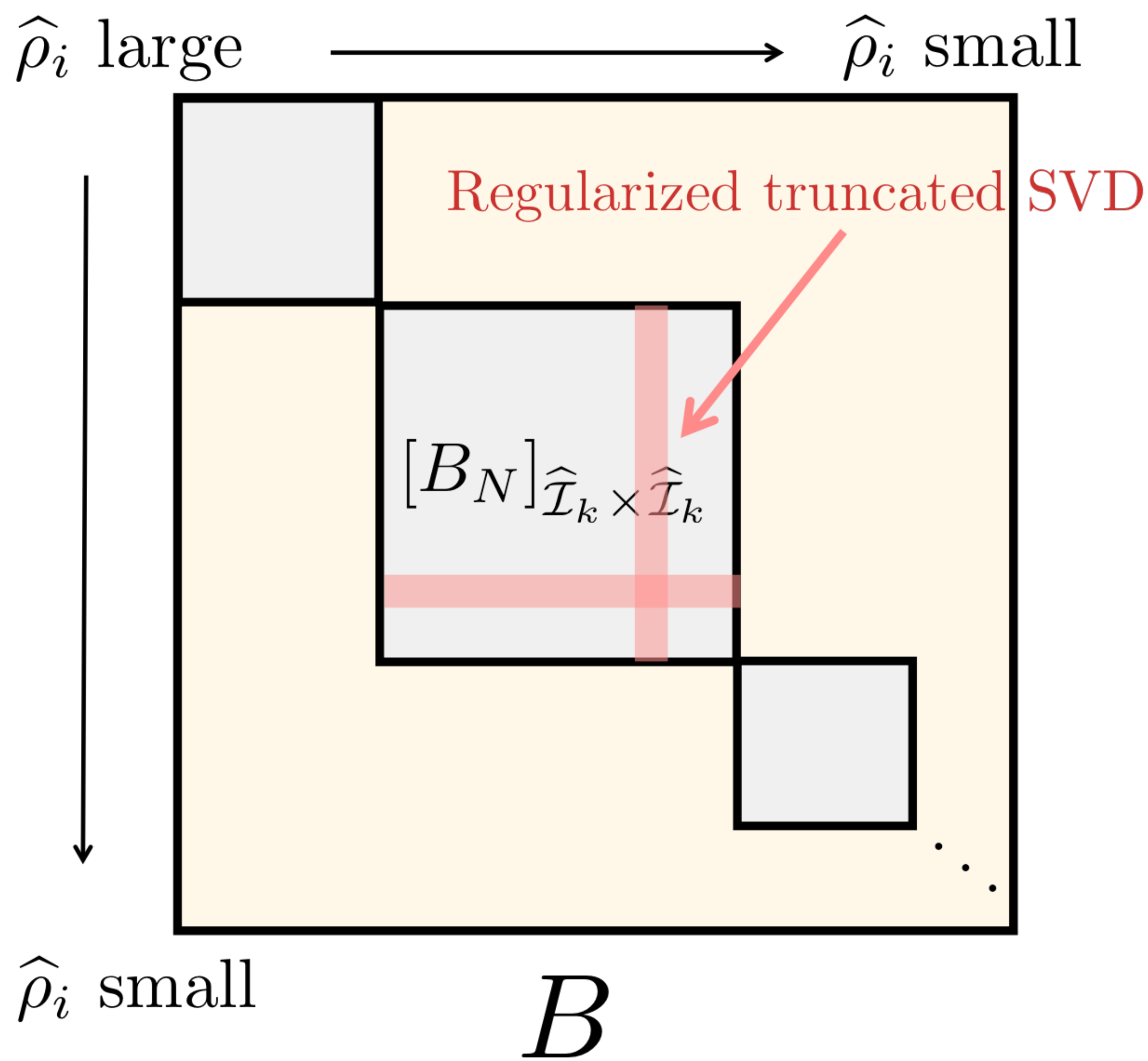
Key Challenge:
In our general setup, we do not have homogeneous marginal probabilities

Algorithmic Idea 1, Binning

We group words according to the empirical marginal probability, divide the matrix to blocks, then apply regularized t-SVD to each block

Phase 1

1. Estimate non-uniform marginal $\widehat{\rho}$
2. Bin M words according to $\widehat{\rho}_i$
3. Regularized t-SVD in each bin × bin block of B to estimate



Key Challenges:

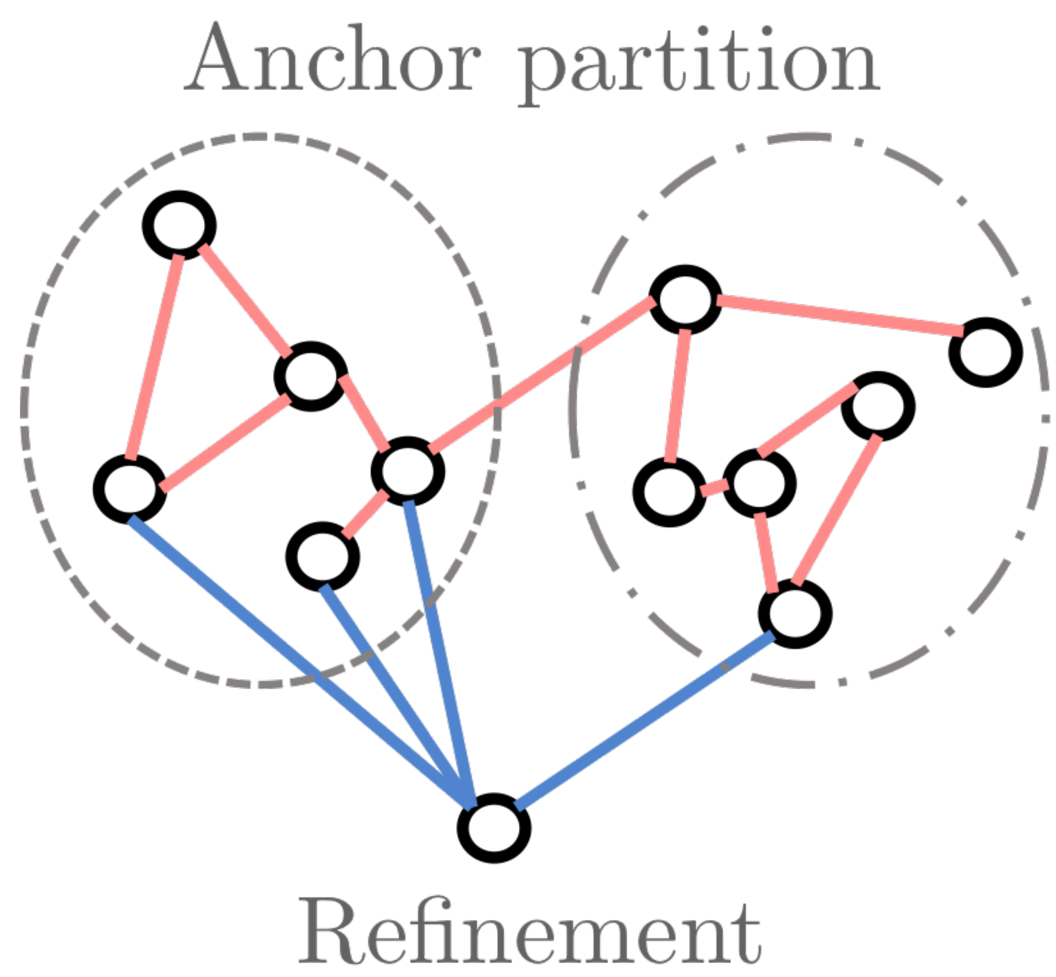
- ✓ Binning is not exact, we need to deal with spillover!
- ✓ We need to piece together estimates over bins!

2. Low rank matrix Algorithmic Idea 2, Refinement

The coarse estimation from Phase 1 gives some global information
Make use of that to do local refinement for each row / column

Phase 2

1. Phase 1 gives coarse estimate for many words
2. Refine the estimate for each word use linear regression
3. Achieve sample complexity $N = O(M/\epsilon^2)$



Conclusion

- Spectral methods are powerful tools for learning mixture models. We can go beyond worst case analysis by exploiting the randomness in the analysis / algorithm.

- To make spectral algorithms more practical, one needs careful algorithm implementation to improve sample complexity.

Future works

- ✓ **Robustness:**
  Agnostic learning, generalization error analysis…
- ✓ **Dynamics:**
  Extend the analysis techniques and algorithmic ideas to learning of random processes, with streaming data, iterative algorithms…
- ✓ **Get hands dirty with real $X(\theta)$!**

References

- “Learning Mixture of Gaussians in High dimensions”
  R. Ge, **H**, S. Kakade (STOC 2015)

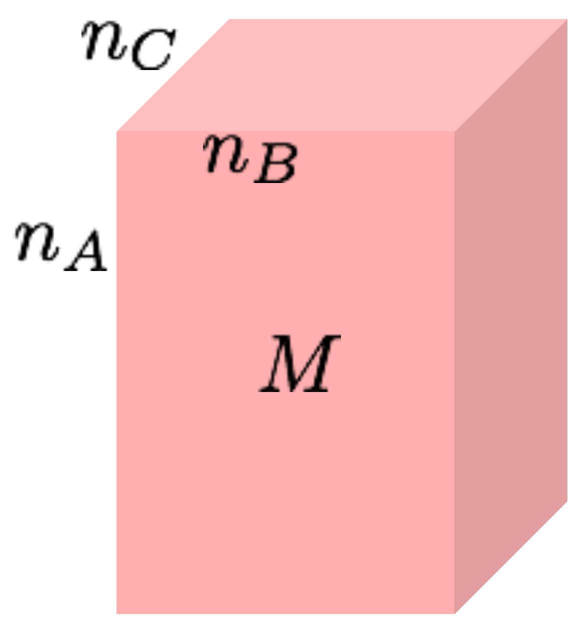
- “Super-Resolution off the Grid”
  H, S. Kakade (NIPS 2015)

- “Minimal Realization Problems for Hidden Markov Models”
  H, R. Ge, S. Kakade, M. Dahleh (IEEE Transactions on Signal Processing, 2016)

- “Recovering Structured Probability Matrices “
  H, S. Kakade, W. Kong, G. Valiant, (submitted to STOC 2016)

Tensor Decomposition

- Multi-way array in matlab
- 2-way tensor =matrix
- 3-way tensor:



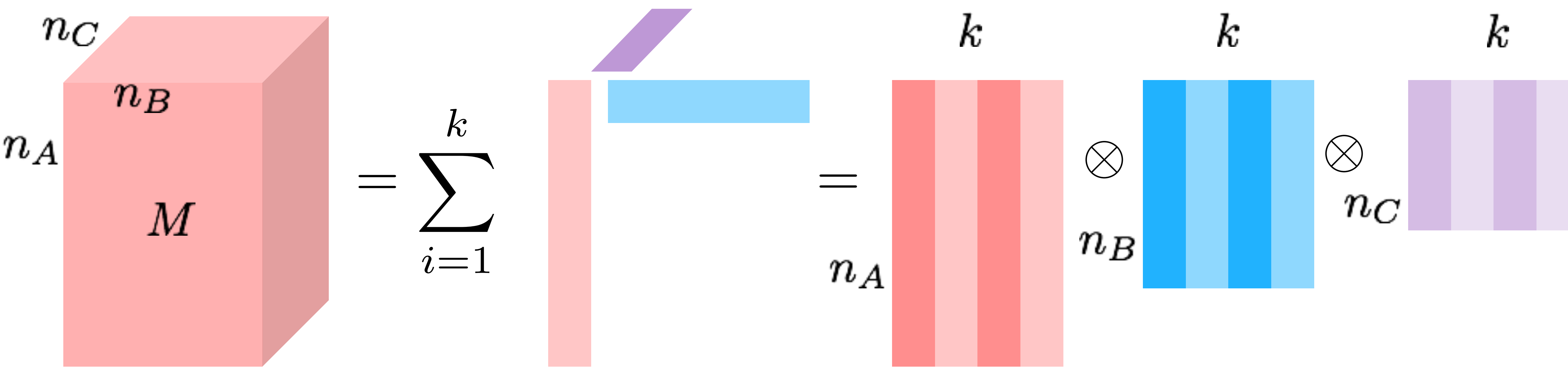
$$M_{j_1, j_2, j_3}, \quad j_1 \in [n_A], j_2 \in [n_B], j_3 \in [n_C]$$

Tensor Decomposition

- Sum of rank one tensors $[\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}]_{j_1, j_2, j_3} = a_{j_1} b_{j_2} c_{j_3}$

$$M = \sum_{i=1}^{k} A_{[:,i]} \otimes B_{[:,i]} \otimes C_{[:,i]} = A \quad \otimes \quad B \quad \otimes \quad C$$

Tensor rank: minimum number of summands in a rank decomposition



Tensor Decomposition

$$M = \sum_{i=1}^{k} A_{[:,i]} \otimes B_{[:,i]} \otimes C_{[:,i]} = A \quad \otimes \quad B \quad \otimes \quad C$$

- Necessary condition for unique tensor decomposition

  If A and B are of full rank k, and C has rank ≥ 2

  we can decompose M to uniquely find the factors $A\ B\ C$

  in poly time and stability depends poly on condition number of $A\ B\ C$

  (the algorithm boils down to matrix SVD)