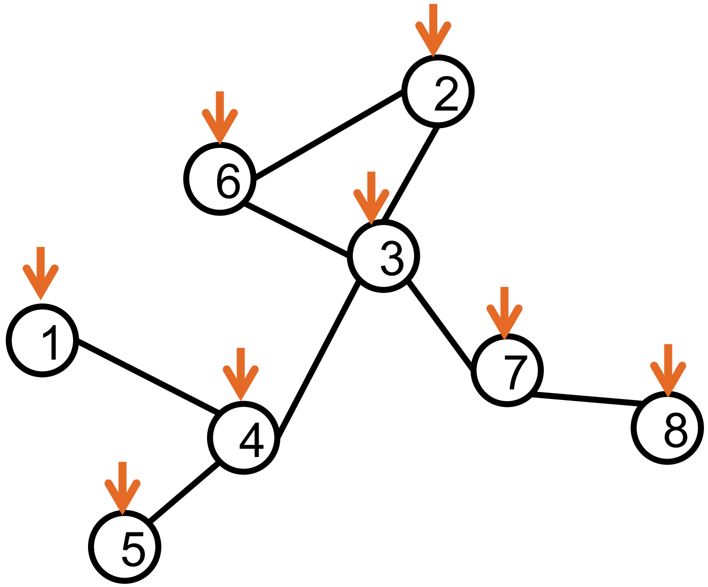
Dynamic Network Volatility



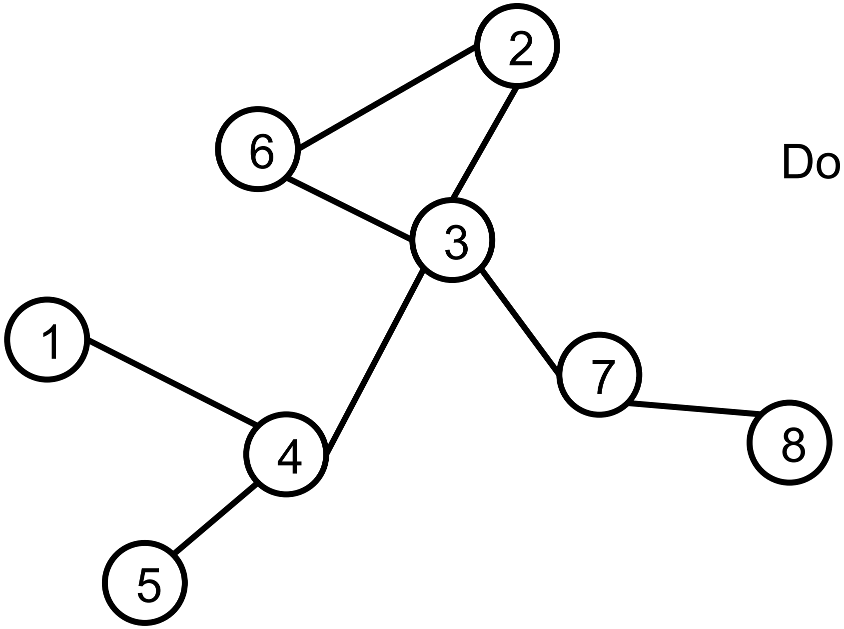
Qingqing Huang (MIT)
collaborated with Ye Yuan (Cambridge University)
Jorge Gonçalves (Cambridge University) Munther Dahleh (MIT)

Informs October 9, 2014

MOTIVATION

Opinion network

Do people like it?



MOTIVATION

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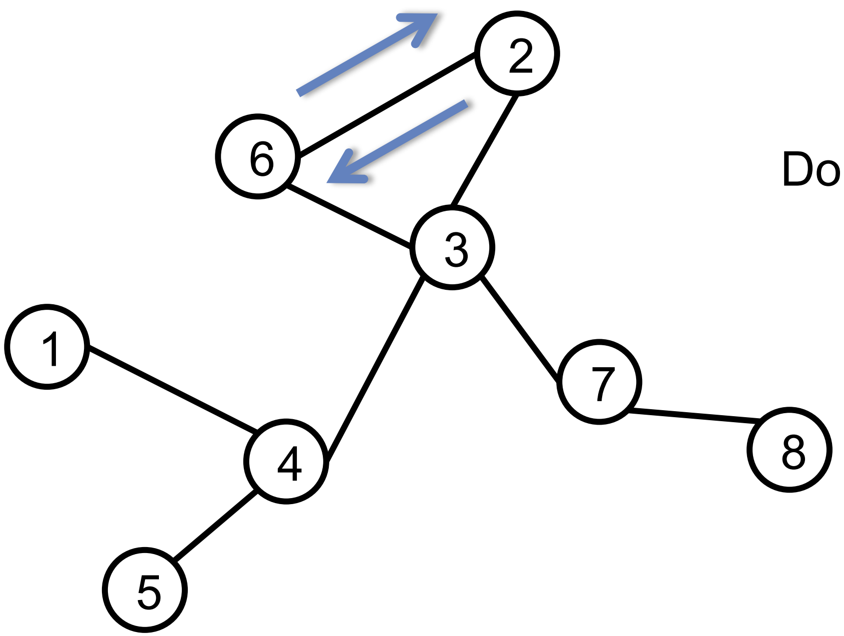
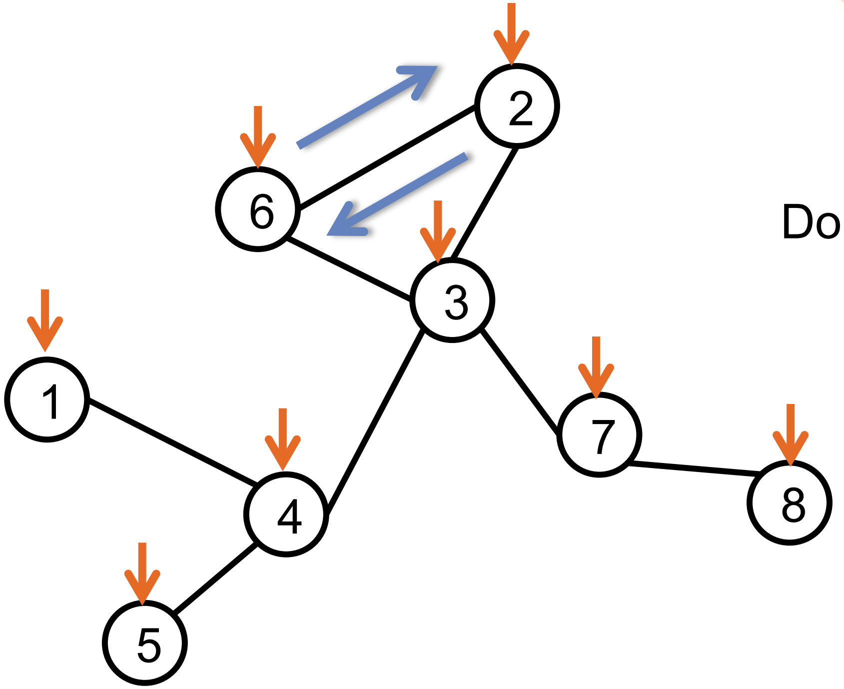


MOTIVATION

Opinion network

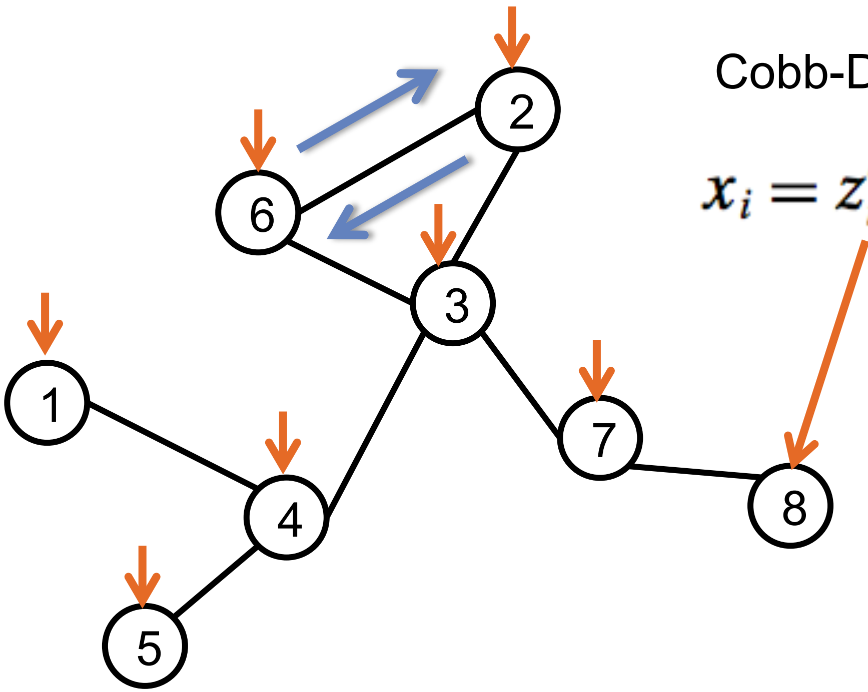
Which consumers to survey?



Do people like it?

MOTIVATION

Economic input-output network



Cobb-Douglas production

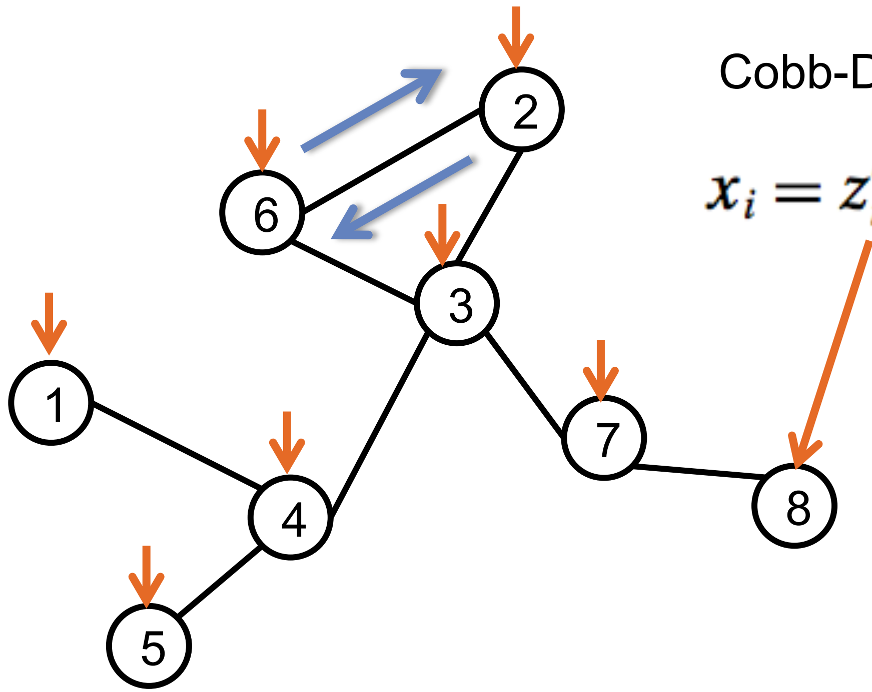
$$x_i = z_i^{\alpha} \ell_i^{\alpha} \prod_{j=1}^{n} x_{ij}^{(1-\alpha)w_{ij}}$$

MOTIVATION

Economic input-output network

Which sectors are more volatile?

Which sectors amplify local shocks the most?



Cobb-Douglas production

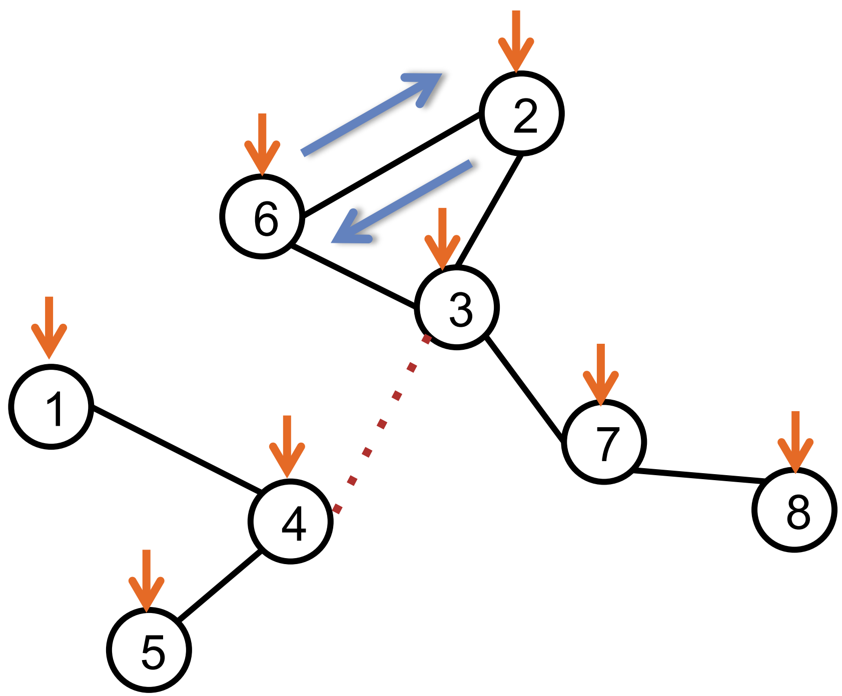
$$x_i = z_i^{\alpha} \ell_i^{\alpha} \prod_{j=1}^{n} x_{ij}^{(1-\alpha)w_{ij}}$$

MOTIVATION

Noisy sensor fusion

Which nodes to measure?

Which communication links are critical?

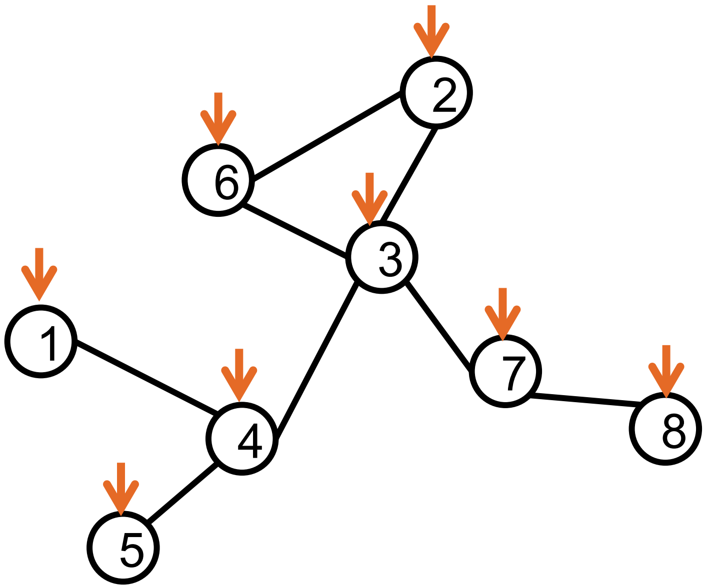


$$L = \Delta - A$$

$$\dot{x} = -Lx$$

QUESTIONS

- How to measure volatility?
- Which networks are more volatile?
- Which nodes are more robust?
- Which links are critical?



SETUP

Stable A

$$\dot{x}(t) = Ax(t) + w(t), \quad t \in [0, +\infty)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & & \\ \vdots & & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \ldots & A_{n,n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \vdots \\ \omega_n(t) \end{bmatrix}$$

State (random process)

Network

Noise (random process)

State fluctuation ← **Noise**

SETUP

Stable A

$$\dot{x}(t) = Ax(t) + w(t), \quad t \in [0, +\infty)$$

- **Measure volatility using H2 norm**
  - Variance amplification $Var(\omega(t)) \rightarrow Var(x(t))$
  - Threshold $\omega(t) \in \mathcal{L}_2 \quad \rightarrow \quad x(t) \in \mathcal{L}_\infty$

SETUP Stable A

$$\dot{x}(t) = Ax(t) + w(t), \quad t \in [0, +\infty)$$

- **How to compute H2 norm**

$$\mathbb{E}[w(t)w(t)'] = W\delta(t - t'),$$

$$AP + PA' + W = 0.$$

State covariance matrix (P)

Noise covariance matrix (W)

SETUP

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$$\mathbb{E}[w(t)w(t)'] = W\delta(t - t'),$$

I.I.D noise: $W = I$

Aggregate noise: $W = \mathbf{1}\mathbf{1}'$

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Noise covariance matrix

I.I.D noise: $W = I$

Nodal Volatility $v_k^{iid} = P_{k,k}$

Network Volatility $V^{iid} = \sum_k v_k^{iid}$

Aggregate noise: $W = \mathbf{1}\mathbf{1}'$

Nodal Volatility $v_k^{agg} = P_{k,k}$

Network Volatility $V^{agg} = \sum_k v_k^{iid}$

- **How to measure volatility?**

- New volatility measures V^{iid} v_k^{iid} V^{agg} v_k^{agg}

- A new measure for link criticality $\frac{\partial V^{iid}}{\partial A_{k,k'}}$:

SPECTRAL PROPERTIES

Stable, symmetric A $A = U\Lambda U'$

	I.I.D noise	Aggregate noise
Nodal volatility	$v_k^{iid} = -\frac{1}{2}\sum_{i=1}^{n}\frac{U_{k,i}^2}{\lambda_i},$	$v_k^{agg} = -\sum_{1\leq i,j\leq n}\frac{U_{k,i}U_{k,j}r_ir_j}{\lambda_i+\lambda_j},$
Network volatility	$V^{iid} = -\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}.$	$V^{agg} = -\frac{1}{2}\sum_{i=1}^{n}\frac{r_i^2}{\lambda_i},$

$$r_i = 1'U_{\cdot,i}$$

SPECTRAL PROPERTIES

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Link criticality of $A_{k,k'}$

$$\frac{\partial V^{iid}}{\partial A_{k,k'}} = \sum_{i=1}^{n}\frac{U_{k,i}U_{k',i}}{\lambda_i^2}$$

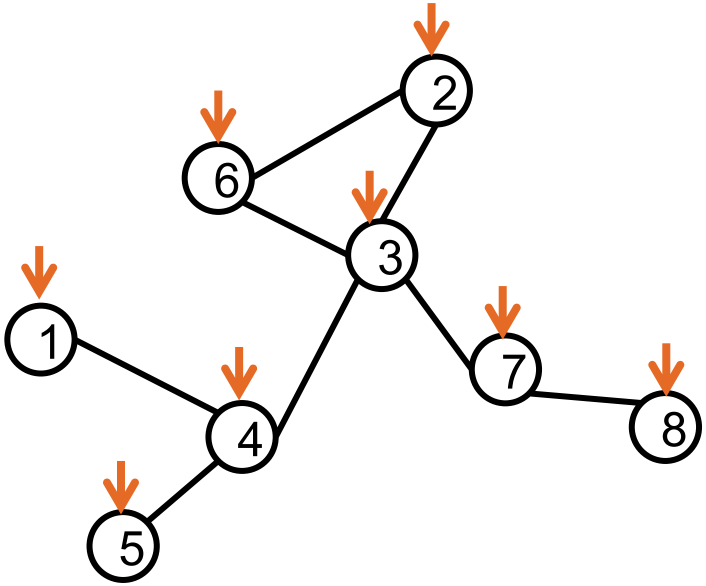
GRAPH-BASED PROPERTIES I

- If the linear dynamics is first-order noisy consensus

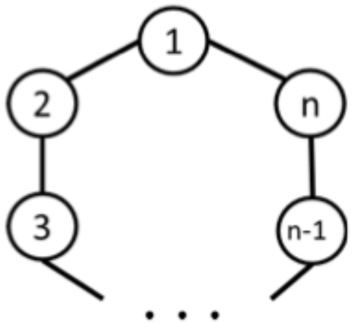
$$\dot{x}(t) = Ax(t) + w(t), \quad t \in [0, +\infty)$$
$$A = -(D - \mathbb{A}).$$

Network Linear Dynamics

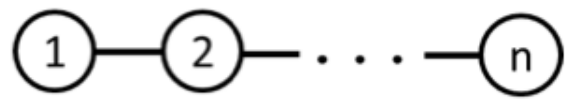
Graph Laplacian Matrix



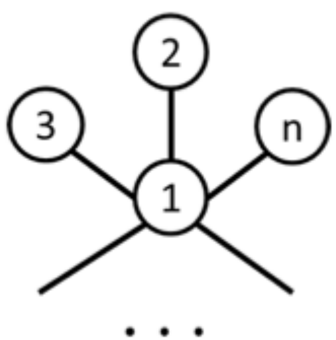
FUNDAMENTAL GRAPHS



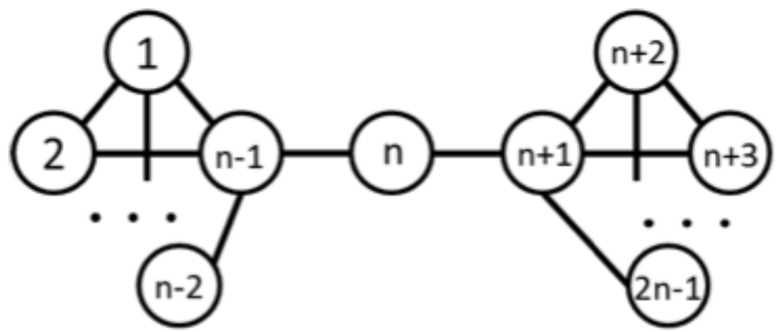
(a) A ring R_n



(b) A chain C_n

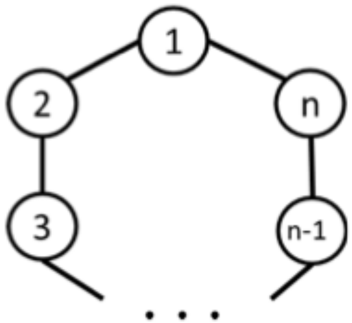


(c) A star S_n



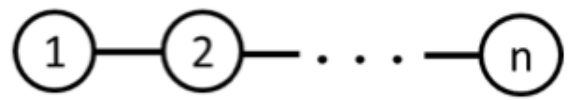
(d) 2-clique graph M_{2n-1}

FUNDAMENTAL GRAPHS



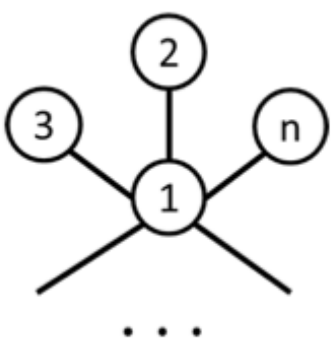
(a) A ring R_n

V^{iid} $\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{1-\cos\frac{2\pi i}{n}}$



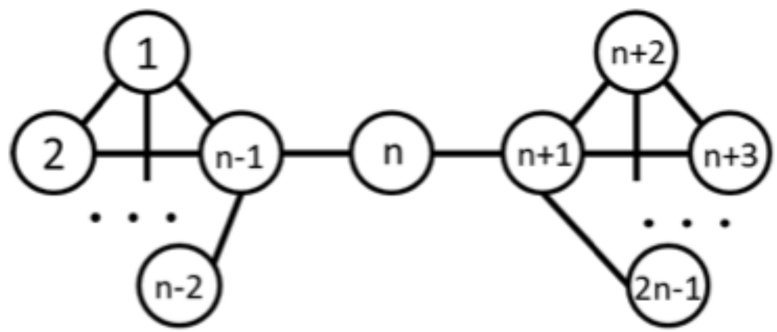
(b) A chain C_n

$\sum_{i=1}^{n-1}\frac{1}{1-\cos\frac{2\pi i}{n}}$



(c) A star S_n

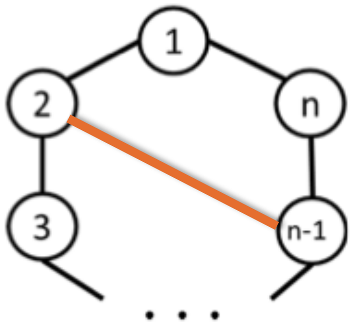
$n+\frac{1}{n}-2$



(d) 2-clique graph M_{2n-1}

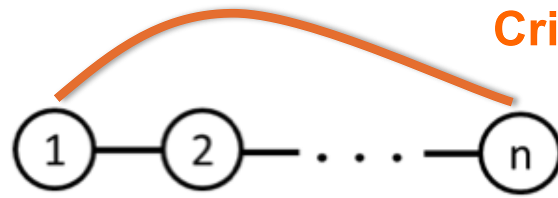
$n+\frac{n+2}{2n-1}+\frac{2n-6}{n-1}$

FUNDAMENTAL GRAPHS



(a) A ring R_n

Critical links

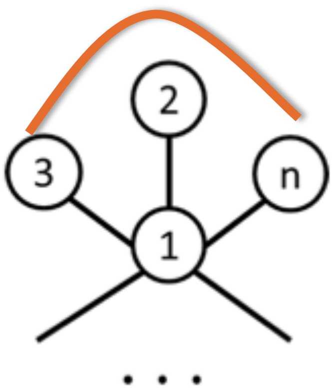


(b) A chain C_n

V^{iid}

$\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{1-\cos\frac{2\pi i}{n}}$

$\sum_{i=1}^{n-1}\frac{1}{1-\cos\frac{2\pi i}{n}}$



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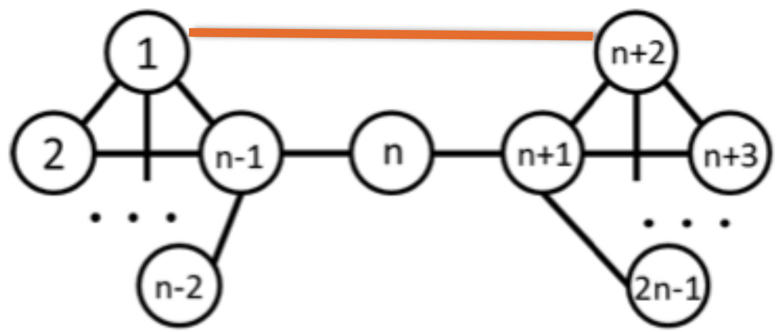


(d) 2-clique graph M_{2n-1}

$n+\frac{n+2}{2n-1}+\frac{2n-6}{n-1}$

For network system with first order linear dynamics

- **Which types of networks are more volatile?**
  - A has small stability margin

$$V^{iid} = -\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}.$$

- **Which nodes are more robust?**
  - ≠ degree, criticality

$$v_k^{iid} = -\frac{1}{2}\sum_{i=1}^{n}\frac{U_{k,i}^2}{\lambda_i},$$

- **Which links are critical?**
  - ≠ betweenness

$$\frac{\partial V^{iid}}{\partial A_{k,k'}} = \sum_{i=1}^{n}\frac{U_{k,i}U_{k',i}}{\lambda_i^2}$$

GRAPH-BASED PROPERTIES II

- If the linear dynamics is first-order consensus

$$A = -(D - \mathbb{A}).$$

Network Linear Dynamics → A

Graph Laplacian Matrix → $D - \mathbb{A}$

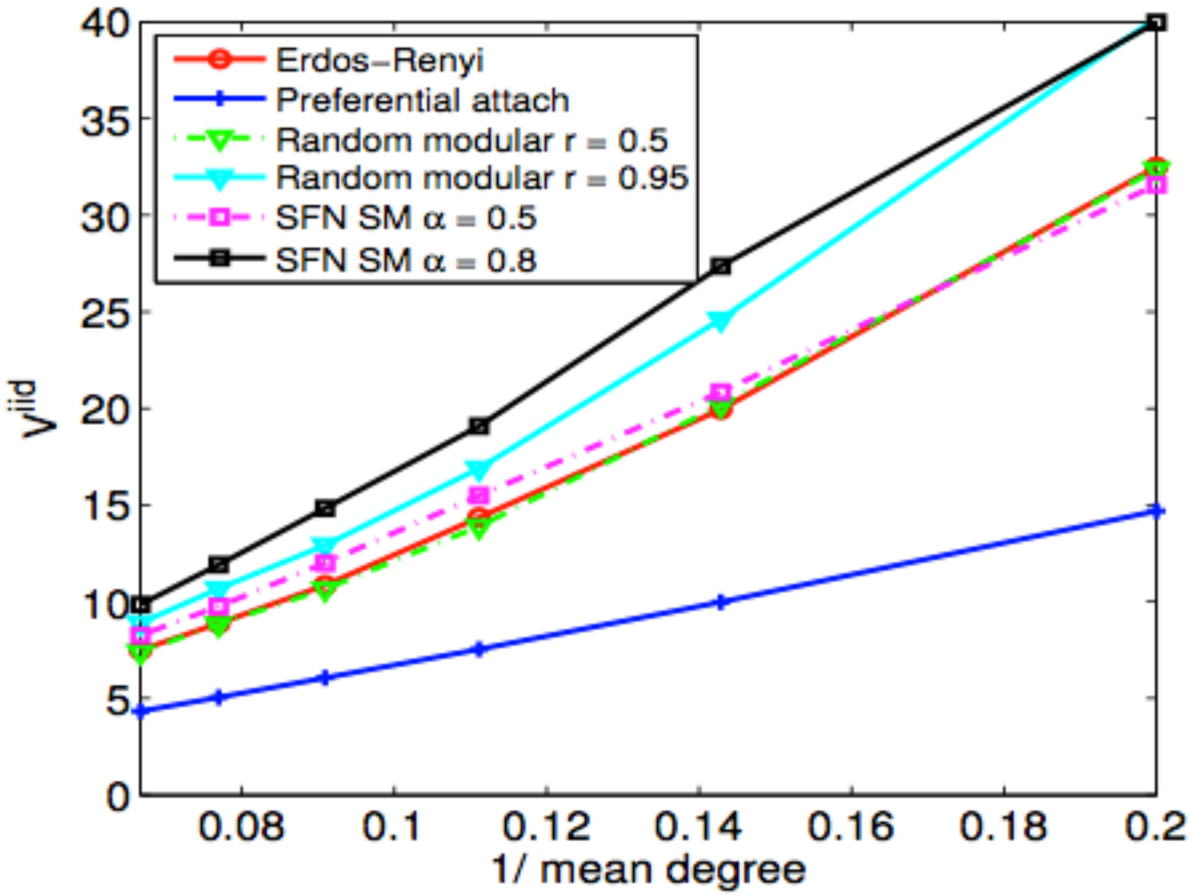
- Degree based approximations of volatility measures

$$v_k^{iid} \geq \frac{1}{2d_k} + \left(\frac{1}{2d_k} - \frac{1}{n}\right) \frac{\sum_{k' \in \mathcal{N}(k)} \frac{1}{d_{k'}}}{|\mathcal{N}(k)|}$$

$$\frac{\partial V^{iid}}{\partial \mathbb{A}_{k,k'}} \approx -\frac{1}{d_k^2}\left(1 - \frac{2\mathbb{A}_{k,k'}}{d_{k'}}\right) - \frac{1}{d_{k'}^2}\left(1 - \frac{2\mathbb{A}_{k,k'}}{d_k}\right)$$

RANDOM NETWORKS

V^{iid} VS 1/ (mean degree)



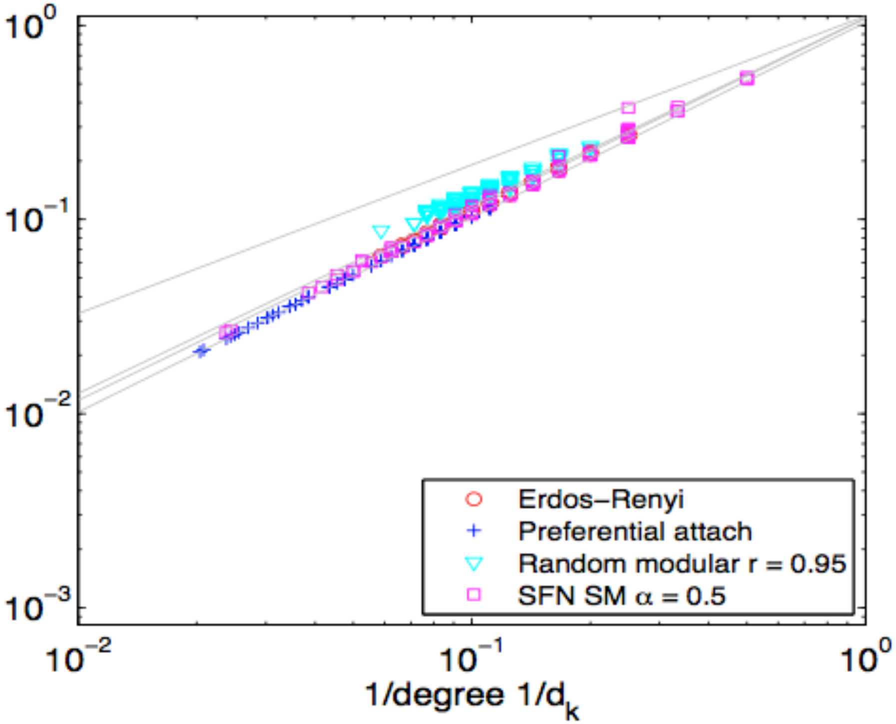
V^{iid} VS

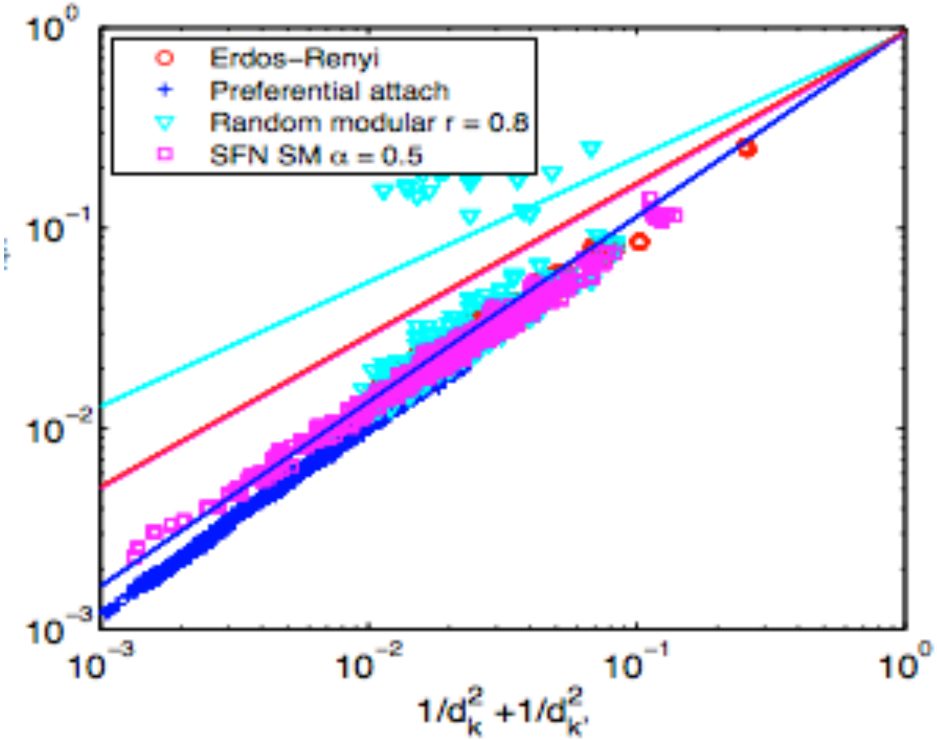
- Mean degree
- Graph modularity
- Degree distribution (power law)

RANDOM NETWORKS

v_k^{iid} VS $\frac{1}{d_k}$

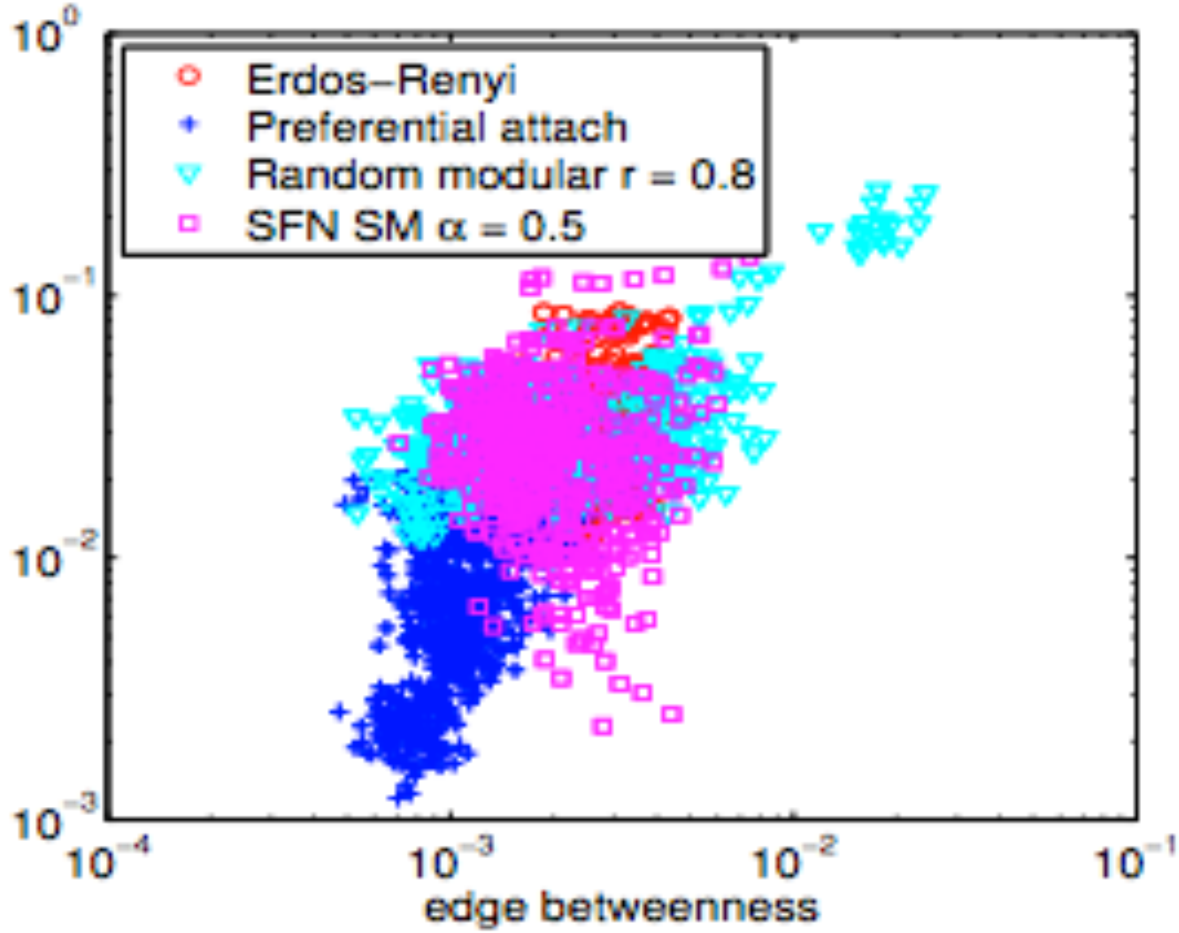
$\frac{\partial V^{iid}}{\partial A_{k,k'}}$ VS $\frac{1}{d_k^2} + \frac{1}{d_{k'}^2}$





RANDOM NETWORKS

$\frac{\partial V^{iid}}{\partial A_{k,k'}}$: **VS betweenness**



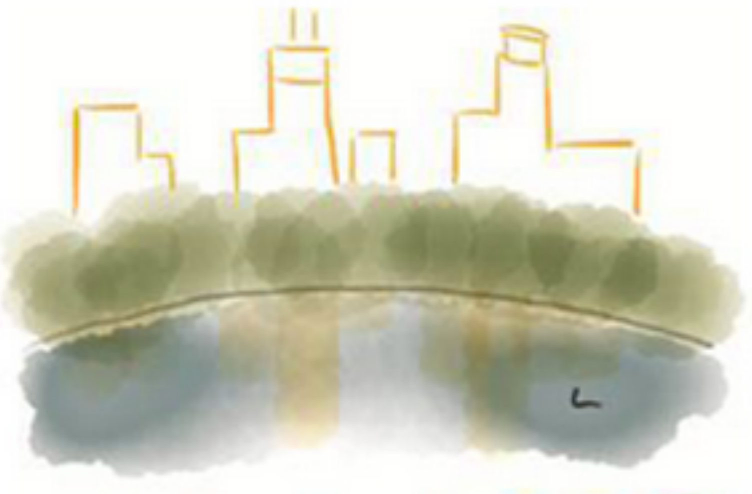
For first order consensus dynamics

- **Which types of networks are more volatile?**
  - ≈ Heterogeneous degree distribution,
  - ≈ high modularity,
  - ≈ loosely connected
- **Which nodes are more robust?**
  - ≈ Hubs
- **Which links are critical?**
  - ≈ Information centrality,
  - ≈ betweenness

CONCLUSION

- **Graph based "centrality measures" do not lead to meaningful implications for real dynamics over networks.**
- **One should examine the real dynamics to measure network robustness / volatility, link criticality.**

- **For linear dynamics over networks, we propose volatility measures which can offer guidance to network design.**
- **For consensus dynamics, we establish the relations between the proposed measures and other graph based properties.**



THANK YOU